

A VARIATIONAL REPRESENTATION AND LARGE DEVIATIONS FOR FUNCTIONALS OF G -BROWNIAN MOTION

FUQING GAO

ABSTRACT. A variational representation for functionals of G -Brownian motion is established by a finite-dimensional approximate technique. As an application of the variational representation, we obtain a large deviation principle for stochastic flows driven by G -Brownian motion.

1. INTRODUCTION

Peng ([20]) proposed G -Brownian motion and G -expectation. The stochastic analysis under the G -expectation (G -stochastic calculus) has had many important progresses in recent years (cf. [22] and references therein). For a connection between Denis and Martini ([9]) and the G -stochastic integration theory of Peng ([20]), we refer to Denis, Hu and Peng ([10]), and Soner, Touzi and Zhang ([25]). The G -stochastic calculus also provides a framework for financial problems with uncertainty about the volatility and a stochastic method for fully nonlinear PDEs (cf. [9],[20], [26]).

The purpose of this paper is to establish a variational representation for functionals of G -Brownian motion and a large deviation principle for stochastic flows driven by G -Brownian motion. We obtain the following variational representation for functionals of G -Brownian motion:

$$\mathbb{E}^G(e^{\Phi(B)}) = \exp \left\{ \sup_{\eta \in (M^2(0,T))^d} \mathbb{E}^G(\Phi(B^\eta) - H_T^G(\eta)) \right\}, \quad (1.1)$$

where $\Phi \in L_G^1(\Omega_T)$ bounded, $\{B_t, t \in [0, T]\}$ is a G -Brownian motion and $\{\langle B \rangle_t, t \in [0, T]\}$ is its quadratic variation process, $B_t^\eta = B_t + \int_0^t \eta_s d\langle B \rangle_s$ and $H_T^G(\eta) = \frac{1}{2} \sum_{i,j=1}^d \int_0^T \eta_s^i \eta_s^j d\langle B \rangle_s^{ij}$. The definitions of \mathbb{E}^G , $L_G^1(\Omega_T)$, $M^2(0, T)$, the G -Brownian motion and the quadratic variation process will be given in Section 2. As an application of the variational representation, we obtain a large deviation principle for stochastic flows driven by G -Brownian motion.

In the classical case, a variational representation of functionals of finite dimensional Brownian motion was first obtained by Boué and Dupuis ([4]). Chen and

2000 *Mathematics Subject Classification.* 60J65, 60F10, 60H10.

Key words and phrases. Variational representation, G -Brownian motion, stochastic flows, large deviations.

Research supported by the National Natural Science Foundation of China (11171262) .

Xiong ([7]) considered the variational representations under a g -expectation which is defined by a backward stochastic differential equation. The variational representations have been shown to be useful in deriving various asymptotic results in large deviations (cf. [4], [5], [6], [11] and [23]) and functional inequalities (cf. [3]).

Under G -expectation, the complicated measurable selection technique in [4] cannot be used and the Clark-Ocone formula is not available. In this paper, we will develop finite-dimensional approximate technique under G -expectation. We prove that a finite-dimensional functional for G -Brownian motion can be approximated by a sequence of G -stochastic differential equations, which plays an important role in the proof of the upper bound. The lower bound will be proved by the G -Girsanov transformation (cf. [29]) and bounded approximation. In particular, this also provides a new proof for the variational representations of Boué and Dupuis.

The remainder of the paper is organized as follows. In Section 2 we recall some basic conceptions and results under G -framework. The variational representation is proved in Section 3. An abstract large deviation principle for functionals of G -Brownian motion is presented in Section 4. A large deviation principle of stochastic flows driven by G -Brownian motion is established in Section 5.

2. G -EXPECTATION AND G -BROWNIAN MOTION

In this section, we briefly recall some basic conceptions and results about G -expectation and G -Brownian motion (see [10],[20], [21] and [22] for details).

2.1. Sublinear expectation. Let Ω be a given set and let \mathcal{H} be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c , and satisfying: if $X_i \in \mathcal{H}$, $i = 1, \dots, d$, then

$$\varphi(X_1, \dots, X_d) \in \mathcal{H}, \text{ for all } \varphi \in \text{lip}(\mathbb{R}^d),$$

where $\text{lip}(\mathbb{R}^d)$ is the space of all bounded and Lipschitz continuous functions on \mathbb{R}^d .

A sublinear expectation $\hat{\mathbb{E}}$ on \mathcal{H} is a functional $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties:

Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}(X) \geq \hat{\mathbb{E}}(Y)$;

Constant preserving: $\hat{\mathbb{E}}(c) = c$;

Sub-additivity: $\hat{\mathbb{E}}(X) - \hat{\mathbb{E}}(Y) \leq \hat{\mathbb{E}}(X - Y)$;

Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}(X)$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in (Ω, \mathcal{H}) .

A m -dimensional random vector $X = (X_1, \dots, X_m)$ is said to be independent of another n -dimensional random vector $Y = (Y_1, \dots, Y_n)$ if

$$\hat{\mathbb{E}}(\varphi(X, Y)) = \hat{\mathbb{E}}(\hat{\mathbb{E}}(\varphi(X, y))_{y=Y}), \text{ for } \varphi \in \text{lip}(\mathbb{R}^m \times \mathbb{R}^n).$$

Let X_1 and X_2 be two d -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically

distributed, denoted by $X_1 \sim X_2$, if

$$\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)), \quad \forall \varphi \in C_{b,Lip}(\mathbb{R}^n).$$

A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G -normal distributed if for each $a, b \geq 0$ we have $aX + b\bar{X} \sim \sqrt{a^2 + b^2}X$, where \bar{X} is an independent copy of X . The letter G denotes the function $G : \mathbb{S}_d \mapsto \mathbb{R}$: $G(A) := \frac{1}{2}\hat{\mathbb{E}}((AX, X))$, where \mathbb{S}_d is the collection of $d \times d$ symmetric matrices and (\cdot, \cdot) denotes the inner product in \mathbb{R}^d , i.e., $(x, y) := \sum_{i=1}^d x^i y^i$ for any $x = (x^1, \dots, x^d)$, $y = (y^1, \dots, y^d) \in \mathbb{R}^d$.

Let d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be G -normal distributed. For each $\varphi \in lip(\mathbb{R}^d)$, set $u(t, x) = \hat{\mathbb{E}}(\varphi(x + \sqrt{t}X))$, $t \geq 0$, $x \in \mathbb{R}^d$. Then $u(t, x)$ is the unique viscosity solution of the following equation,

$$\frac{\partial u}{\partial t} - G(D_x^2 u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad u(0, x) = \varphi(x), \quad (2.1)$$

where $D_x^2 u = (\partial_{x^i x^j}^2 u)_{i,j=1}^d$ is the Hessian matrix of u .

The inner product in \mathbb{S}_d is defined by $(A_1, A_2) = \sum_{i,j=1}^d a_1(i, j)a_2(i, j)$ for $A_1 = (a_1(i, j))_{d \times d}$, $A_2 = (a_2(i, j))_{d \times d}$. Then the map $G : \mathbb{S}_d \mapsto \mathbb{R}$ is a monotonic and sublinear function, i.e., for $A, \bar{A} \in \mathbb{S}_d$,

$$\begin{cases} G(A + \bar{A}) \leq G(A) + G(\bar{A}), \\ G(\lambda A) = \lambda G(A), \quad \text{for all } \lambda \geq 0, \\ G(A) \geq G(\bar{A}), \quad \text{if } A \geq \bar{A}. \end{cases} \quad (2.2)$$

For a monotonic and sublinear function $G : \mathbb{S}_d \mapsto \mathbb{R}$ given, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_d^+$ (the non-negative elements of \mathbb{S}_d) such that $G(A) = \frac{1}{2} \sup_{\sigma \in \Sigma} (A, \sigma)$. Throughout this paper, we assume that there exist constants $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$ such that

$$\Sigma \subset \{\sigma \in \mathbb{S}_d; \underline{\sigma} I_{d \times d} \leq \sigma \leq \bar{\sigma} I_{d \times d}\}. \quad (2.3)$$

2.2. G -Brownian motion and G -expectation. Let Ω denote the space of all \mathbb{R}^d -valued continuous paths $\omega : (0, +\infty) \ni t \mapsto \omega_t \in \mathbb{R}^d$, with $\omega_0 = 0$. Let $\mathcal{B}(\Omega)$, \mathcal{M} , $L^0(\Omega)$, $B_b(\Omega)$ and $C_b(\Omega)$ denote respectively the Borel σ -algebra of Ω , the collection of all probability measure on Ω , the space of all $\mathcal{B}(\Omega)$ -measurable real functions, all bounded elements in $L^0(\Omega)$ and all continuous elements in $B_b(\Omega)$. For each $t \in [0, \infty)$, we also denote $\Omega_t := \{\omega_{\cdot \wedge t} : \omega \in \Omega\}$; $\mathcal{F}_t := \mathcal{B}(\Omega_t)$; $L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$ -measurable real functions; $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$; $C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$.

For each $t > 0$, set

$$Lip(\Omega_t) := \{\varphi(\omega_{t_1}, \omega_{t_2}, \dots, \omega_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, t], \varphi \in lip((\mathbb{R}^d)^n)\}.$$

Define $Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n) \subset C_b(\Omega)$.

Let $G : \mathbb{S}_d \mapsto \mathbb{R}$ be a given monotonic and sublinear function. A continuous process $\{B_t(\omega)\}_{t \geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E}^G)$ is called a G -Brownian motion if it has stationary and independent increments, $B_0 = 0$, B_1 is G -normal distributed and $\mathbb{E}^G(B_t) = -\mathbb{E}^G(-B_t) = 0$ for $t \geq 0$.

The topological completion of $L_{ip}(\Omega_t)$ (resp. $L_{ip}(\Omega)$) under the Banach norm $\|\cdot\|_{p,G} := (\mathbb{E}^G(|\cdot|^p))^{1/p}$ is denoted by $L_G^p(\Omega_t)$ (resp. $L_G^p(\Omega)$), where $p \geq 1$. $\mathbb{E}^G(\cdot)$ can be extended uniquely to a sublinear expectation on $L_G^1(\Omega)$. We denote also by \mathbb{E}^G the extension. It is proved in [10] that $L^0(\Omega) \supset L_G^p(\Omega) \supset C_b(\Omega)$, and there exists a weakly compact family \mathcal{P} of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that $\mathbb{E}^G(X) = \sup_{P \in \mathcal{P}} E_P(X)$ for $X \in C_b(\Omega)$. $\mathbb{E}^G(\cdot)$ has the following regularity ([10]): For each $\{X_n\}_{n=1}^\infty$ in $C_b(\Omega)$ with $X_n \downarrow 0$ on Ω , $\mathbb{E}^G(X_n) \downarrow 0$. We also denote

$$\overline{\mathbb{E}}^G(X) = \sup_{P \in \mathcal{P}} E_P(X), \quad X \in L^0(\Omega).$$

The natural Choquet capacity associated with \mathbb{E}^G is defined by

$$c^G(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

A set $A \subset \Omega$ is polar if $c^G(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. A mapping X on Ω with values in a topological space is said to be quasi-continuous (q.c.) if for any $\varepsilon > 0$, there exists an open set O with $c^G(O) < \varepsilon$ such that $X|_{O^c}$ is continuous. $L_G^p(\Omega)$ also has the following characterization ([10]):

$$L_G^p(\Omega) = \left\{ X \in L^0(\Omega); \lim_{n \rightarrow \infty} \overline{\mathbb{E}}^G(|X|^p I_{\{|X| \geq n\}}) = 0, \right. \\ \left. \text{and } X \text{ is } c^G\text{-quasi surely continuous} \right\}.$$

Let us recall the representation theorem of G -expectation. For a monotonic and sublinear function $G : \mathbb{S}_d \mapsto \mathbb{R}$ given, there exists a bounded, convex and closed subset $\Sigma \subset \mathbb{S}_d^+$ such that $G(A) = \frac{1}{2} \sup_{\sigma \in \Sigma} (A, \sigma)$. Set $\Gamma := \{\gamma = \sigma^{1/2}, \sigma \in \Sigma\}$. Let P be the Wiener measure on Ω . Let $\mathcal{A}_{0,\infty}^\Gamma$ be the collection of all Γ -valued $\{\mathcal{F}_t, t \geq 0\}$ -adapted processes on the interval $[0, +\infty)$, i.e., $\{\theta_t, t \geq 0\} \in \mathcal{A}_{0,\infty}^\Gamma$ if and only if θ_t is \mathcal{F}_t measurable and $\theta_t \in \Gamma$ for each $t \geq 0$, and let P_θ be the law of the process $\{\int_0^t \theta_s d\omega_s, t \geq 0\}$ under the Wiener measure P . The representation theorem of G -expectation([10]) is stated as follows: for all $X \in L_G^1(\Omega)$

$$\mathbb{E}^G(X) = \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta}(X). \quad (2.4)$$

2.3. G -Stochastic integral and quadratic variation process. Given $T > 0$. For $p \in [1, \infty)$, let $M_G^{p,0}(0, T)$ denote the space of \mathbb{R} -valued piecewise constant processes

$$\eta_t = \sum_{i=0}^{n-1} \eta_{t_i} 1_{[t_i, t_{i+1})}(t)$$

where $\eta_{t_i} \in L_G^p(\Omega_{t_i})$, $0 = t_0 < t_1 < \dots < t_n = T$. For $\eta \in M_G^{p,0}(0, T)$, $j = 1, \dots, d$, the G -stochastic integral is defined by

$$I^j(\eta) := \int_0^T \eta_s dB_s^j := \sum_{i=0}^{n-1} \eta_{t_i} (B_{t_{i+1}}^j - B_{t_i}^j).$$

Let $M_G^p(0, T)$ be the closure of $M_G^{p,0}(0, T)$ under the norm: $\|H\|_{M_G^p(0, T)}^p := \mathbb{E}^G \left(\int_0^T |\eta_t|^p dt \right)$.

Then the mapping $I^j : M_G^{2,0}(0, T) \rightarrow L_G^2(\Omega_T)$ is continuous, and so it can be continuously extended to $M_G^2(0, T)$.

For any $\eta = (\eta^1, \dots, \eta^d) \in (M_G^2(0, T))^d$, define

$$\int_0^T \eta_s dB_s = \sum_{i=1}^d \int_0^T \eta_s^i dB_s^i.$$

The quadratic variation process of G -Brownian motion is defined by

$$\langle B \rangle_t := (\langle B \rangle_t^{ij})_{1 \leq i, j \leq d} = \left(B_t^i B_t^j - 2 \int_0^t B_s^i dB_s^j \right)_{1 \leq i, j \leq d}, \quad 0 \leq t \leq T.$$

$\langle B \rangle_t$ is a \mathbb{S}_d -valued process with stationary and independent increments.

For any $1 \leq i, j \leq d$, define a mapping $M_G^{1,0}(0, T) \mapsto L_G^T(\Omega_1)$ as follows:

$$Q_{0,T}^{ij}(\eta) = \int_0^T \eta_s d\langle B \rangle_s^{ij} := \sum_{k=0}^{n-1} \eta_{t_k} (\langle B \rangle_{t_{k+1}}^{ij} - \langle B \rangle_{t_k}^{ij}).$$

Then $Q_{0,T}^{ij}$ can be uniquely extended to $M_G^1(0, T)$. We still denote this mapping by

$$\int_0^T \eta_s d\langle B \rangle_s^{ij} = Q_{0,T}^{ij}(\eta), \quad \eta \in M_G^1(0, T).$$

For $\eta = (\eta^1, \dots, \eta^d) \in (M_G^1(0, T))^d$, define

$$\int_0^T \eta_s d\langle B \rangle_s = \left(\sum_{j=1}^d \int_0^T \eta_s^j d\langle B \rangle_s^{ij} \right)_{d \times 1}.$$

and for $\eta = (\eta^{ij})_{d \times d} \in (M_G^1(0, T))^{d \times d}$, define

$$\int_0^T \eta_s d\langle B \rangle_s = \sum_{i,j=1}^d \int_0^T \eta_s^{ij} d\langle B \rangle_s^{ij}.$$

2.4. G -Girsanov formula. In one dimensional case, under a strong Novikov-type condition, Xu, Shang and Zhang ([29]) obtained a Girsanov formula under G -expectation based on the martingale characterization theorem ([28]) of G -Brownian motion. A multi-dimensional version of the G -Girsanov formula is presented in [19].

For $\eta = (\eta^1, \dots, \eta^d) \in (M_G^2(0, T))^d$ satisfying the following strong Novikov condition:

$$\mathbb{E}^G \left(\exp \left\{ \frac{1}{2}(1 + \epsilon) \sum_{i,j=1}^d \int_0^T \eta_s^i \eta_s^j d\langle B \rangle_s^{ij} \right\} \right) < \infty, \quad \text{for some } \epsilon > 0, \quad (2.5)$$

we define

$$\mathcal{E}_t^\eta = \exp \left\{ \int_0^t \eta_s dB_s - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \eta_s^i \eta_s^j d\langle B \rangle_s^{ij} \right\}, \quad 0 \leq t \leq T. \quad (2.6)$$

Then \mathcal{E}_t^η is quasi-continuous. By the condition (2.5), \mathcal{E}_t^η , $t \in [0, T]$ is a martingale under each P_θ , and $\mathcal{E}_t^\eta \in L_G^1(\Omega_t)$, $t \in [0, T]$ (cf. [29]). Set

$$dP_{\theta,\eta} = \mathcal{E}_T^\eta dP_\theta, \quad (2.7)$$

and

$$\mathbb{E}^{G,\eta}(X) = \sup_{\theta \in \mathcal{A}_{0,\infty}^I} P_{\theta,\eta}(X), \quad X \in L_{ip}(\Omega_T). \quad (2.8)$$

Let $L_{G,\eta}^1(\Omega_t)$ be the completion of $(L_{ip}(\Omega_t), \mathbb{E}^{G,\eta}(|\cdot|))$. Then $\mathbb{E}^{G,\eta}$ can be extended to $L_{G,\eta}^1(\Omega_T)$, and the following G-Girsanov formula holds:

G-Girsanov formula. ([29], [19]) Under the condition (2.5),

$$B_t^{-\eta} := B_t - \int_0^t \eta_s d\langle B \rangle_s, \quad t \in [0, T]$$

is a G -Brownian motion under $\mathbb{E}^{G,\eta}$.

3. A VARIATIONAL REPRESENTATION FOR FUNCTIONALS OF G -BROWNIAN MOTION

The main result in this section is the following variation representation for functionals of G -Brownian motion.

Theorem 3.1. *Let $\Phi \in L_G^1(\Omega_T)$ be bounded. Then*

$$\begin{aligned} \mathbb{E}^G(e^{\Phi(B)}) &= \exp \left\{ \sup_{\eta \in (M_G^2(0,T))^d} \mathbb{E}^G(\Phi(B^\eta) - H_T^G(\eta)) \right\} \\ &= \exp \left\{ \sup_{\eta \in (M_G^{2,0}(0,T))^d \cap \mathcal{B}_b(\Omega_T)} \mathbb{E}^G(\Phi(B^\eta) - H_T^G(\eta)) \right\}, \end{aligned} \quad (3.1)$$

where $B_t^\eta := B_t + \int_0^t \eta_s d\langle B \rangle_s$,

$$H_t^G(\eta) = \frac{1}{2} \sum_{i,j=1}^d \int_0^t \eta_s^i \eta_s^j d\langle B \rangle_s^{ij}, \quad t \in [0, T]. \quad (3.2)$$

Remark 3.1. (1). The following proof of Theorem 3.1 is not depend on the representation for ordinary Brownian motion, the variational formula of the relative entropy, the measurable selection technique and the Clark-Ocone formula. In particular, this also gives a new proof of the representation for ordinary Brownian motion in [4].

(2). Notice that the G -expectation is the supremum of a collection of expectations so that the canonical map in Wiener space is a martingale under the expectations. If the representation for ordinary Brownian motion can be extended to continuous martingales, then Theorem 3.1 can be obtained from this extension. But the extension is not available.

Lemma 3.1. (cf. [22], [25]) Let $\eta_s^{ij} \in M_G^2(0, T)$, $\eta_s^{ij} = \eta_s^{ji}$, $i, j = 1, \dots, d$ be given and set

$$M_t = 2 \int_0^t G(\eta_s) ds - \int_0^t \eta_s d\langle B \rangle_s, \quad t \in [0, T].$$

Then $M_t \geq 0$, q.s. for all $t \in [0, T]$. In particular, $t \rightarrow M_t$ is increasing.

Proof. Take a sequence $\eta^{(N)} \in (M_G^{2,0}(0, T))^{d \times d}$, where

$$\eta_s^{(N)} = \sum_{k=1}^N \eta_{t_{k-1}^{(N)}}^{(N)} I_{[t_{k-1}^{(N)}, t_k^{(N)}]}(s), \quad 0 = t_0^{(N)} < t_1^{(N)} < \dots < t_N^{(N)} = T,$$

such that

$$\lim_{N \rightarrow \infty} \max_{1 \leq i, j \leq d} \mathbb{E}^G \left(\int_0^T |(\eta_s^{(N)})^{ij} - \eta_s^{ij}|^2 ds \right) = 0.$$

Then

$$\begin{aligned} & \mathbb{E}^G \left(\int_0^T |G(\eta_s) - G(\eta_s^{(N)})| ds \right) \\ & \leq \mathbb{E}^G \left(\int_0^T \max\{G(\eta_s - \eta_s^{(N)}), G(\eta_s^{(N)} - \eta_s)\} ds \right) \\ & = \frac{1}{2} \mathbb{E}^G \left(\int_0^T \sup_{\sigma \in \Sigma} |(\eta_s - \eta_s^{(N)}, \sigma)| ds \right) \rightarrow 0 \text{ as } N \rightarrow \infty, \end{aligned}$$

which yields that for any $t \in [0, T]$,

$$\lim_{N \rightarrow \infty} \mathbb{E}^G \left(\left| M_t - \left(2 \int_0^t G(\eta_s^{(N)}) ds - \int_0^t \eta_s^{(N)} d\langle B \rangle_s \right) \right| \right) = 0$$

Thus, it is sufficient to consider the case $\eta^{ij} \in M_G^{2,0}(0, T)$, $\eta_s^{ij} = \eta_s^{ji}$, $i, j = 1, \dots, d$, i.e.,

$$\eta_s = \sum_{k=1}^N \eta_{t_{k-1}} I_{[t_{k-1}, t_k]}(s), \quad 0 = t_0 < t_1 < \dots < t_N = T.$$

In this case, we have

$$\begin{aligned}
M_t &= \sum_{k=1}^N (2G(\eta_{t_{k-1}}(t_k - t_{k-1})) - (\eta_{t_{k-1}}, \langle B \rangle_{t_k} - \langle B \rangle_{t_{k-1}})) \\
&= \sum_{k=1}^N \left(\sup_{\sigma \in \Sigma} (\eta_{t_{k-1}}(t_k - t_{k-1}), \sigma) - (\eta_{t_{k-1}}, \langle B \rangle_{t_k} - \langle B \rangle_{t_{k-1}}) \right) \\
&= \sum_{k=1}^N (t_k - t_{k-1}) \left(\sup_{\sigma \in \Sigma} (\eta_{t_{k-1}}, \sigma) - \left(\eta_{t_{k-1}}, \frac{(\langle B \rangle_{t_k} - \langle B \rangle_{t_{k-1}})}{t_k - t_{k-1}} \right) \right) \geq 0.
\end{aligned}$$

□

Proof. The proof of the lower bound of Theorem 3.1.

Step 1. Bounded case. Let η be bounded. Then by the G-Girsanov-formula,

$$\mathbb{E}^G(e^{\Phi(B)}) = \mathbb{E}^{G, -\eta}(e^{\Phi(B^\eta)}) = \mathbb{E}^G \left(e^{\Phi(B^\eta)} \exp \left\{ - \int_0^T \eta_s dB_s - H_T^G(\eta) \right\} \right).$$

By Jensen's inequality, we have

$$\log \mathbb{E}^G(e^{\Phi(B)}) \geq \mathbb{E}^G \left(\Phi(B^\eta) - \int_0^T \eta_s dB_s - H_T^G(\eta) \right) = \mathbb{E}^G (\Phi(B^\eta) - H_T^G(\eta)).$$

Step 2. General case. Choose a sequence $\{\Phi_n, n \geq 1\}$ of uniformly bounded and Lipschitz continuous functions such that

$$\lim_{n \rightarrow \infty} \mathbb{E}^G(|\Phi_n - \Phi|^2) = 0.$$

For $\eta \in (M_G^2(0, T))^d$ given, we can find a sequence $\{\eta^{(m)}, m \geq 1\} \subset (M_G^{2,0}(0, T))^d \cap \mathcal{B}_b(\Omega_T)$ such that

$$\lim_{m \rightarrow \infty} \mathbb{E}^G \left(\int_0^T |\eta_t^{(m)} - \eta_t|^2 dt \right) = 0.$$

Then

$$\lim_{m \rightarrow \infty} \mathbb{E}^G (|H_T^G(\eta^{(m)}) - H_T^G(\eta)|) = 0,$$

and for each $n \geq 1$, by Lipschitz continuity of Φ_n , we also have

$$\lim_{m \rightarrow \infty} \mathbb{E}^G \left(\left| \Phi_n(B^\eta) - \Phi_n(B^{\eta^{(m)}}) \right| \right) \leq l(n, \bar{\sigma}) \lim_{m \rightarrow \infty} \mathbb{E}^G \left(\int_0^T |\eta_t^{(m)} - \eta_t| dt \right) = 0$$

where $l(n, \bar{\sigma})$ is a constant independent of m . Therefore, in order to prove $\mathbb{E}^G(e^{\Phi(B)}) \geq \mathbb{E}^G(\Phi(B^\eta) - H_T^G(\eta))$, it only remains to verify

$$\lim_{n \rightarrow \infty} \sup_{m \geq 1} \mathbb{E}^G \left(\left| \Phi_n(B^{\eta^{(m)}}) - \Phi(B^{\eta^{(m)}}) \right| \right) = 0. \quad (3.3)$$

Fix $\epsilon > 0$. Let $M \in (0, \infty)$ be a uniform upper bound $|\Phi_n|, |\Phi|$, $n \geq 1$. Then

$$\begin{aligned} & \mathbb{E}^G \left(\left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| \right) \\ &= \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(\left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| \right) \\ &\leq 2M \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} P_\theta \left(\left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right) + \epsilon. \end{aligned}$$

Therefore, we only need to show that for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} P_\theta \left(\left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right) = 0. \quad (3.4)$$

For any $N \in (1, \infty)$,

$$\begin{aligned} & \sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} P_\theta \left(\left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right) \\ &= \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} P_\theta \left(I_{\left\{ \left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right\}} \mathcal{E}_T^{-\eta^{(m)}} (\mathcal{E}_T^{-\eta^{(m)}})^{-1} \right) \\ &\leq N \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(I_{\left\{ \left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right\}} \mathcal{E}_T^{-\eta^{(m)}} \right) \\ &\quad + \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(I_{\left\{ \left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right\}} I_{\left\{ \mathcal{E}_T^{-\eta^{(m)}} \leq 1/N \right\}} \right). \end{aligned}$$

By Chebyshev's inequality and the G-Girsanov-formula, for all $m \geq 1$,

$$\begin{aligned} & \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(I_{\left\{ \left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right\}} \mathcal{E}_T^{-\eta^{(m)}} \right) \\ &\leq \frac{1}{\epsilon^2} \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(\left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right|^2 \mathcal{E}_T^{-\eta^{(m)}} \right) \\ &= \frac{1}{\epsilon^2} \mathbb{E}^G (|\Phi_n - \Phi|^2) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

We also have that

$$\begin{aligned} & \sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(I_{\left\{ \left| \Phi_n \left(B^{\eta^{(m)}} \right) - \Phi \left(B^{\eta^{(m)}} \right) \right| > \epsilon \right\}} I_{\left\{ \mathcal{E}_T^{-\eta^{(m)}} \leq 1/N \right\}} \right) \\ &\leq \frac{1}{\log N} \sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}^\Gamma} E_{P_\theta} \left(-\log \mathcal{E}_T^{-\eta^{(m)}} \right) = \frac{1}{\log N} \sup_{m \geq 1} \mathbb{E}^G (H_T^G(\eta^{(m)})) \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

(3.4) is valid. Hence,

$$\mathbb{E}^G(e^{\Phi(B)}) \geq \exp \left\{ \sup_{\eta \in (M_G^2(0,T))^d} \mathbb{E}^G(\Phi(B^n) - H_T^G(\eta)) \right\}. \quad (3.5)$$

The proof of the upper bound Theorem 3.1.

First, let us consider $\Phi = f(B_{t_1}, \dots, B_{t_n})$, where $f \in \text{lip}((\mathbb{R}^d)^n)$ and $0 \leq t_1 < \dots < t_n = T$. Set

$$\|f\| := \sup_{y \in (\mathbb{R}^d)^n} |f(y)|, \quad \|f\|_{\text{lip}} := \sup_{y, z \in (\mathbb{R}^d)^n, y \neq z} \frac{|f(y) - f(z)|}{|y - z|}.$$

We want to show that there exists a constant $C(\|f\|, \|f\|_{\text{lip}}) \in (0, \infty)$ that is only dependent on $\|f\|$ and $\|f\|_{\text{lip}}$, such that

$$\mathbb{E}^G(e^{\Phi(B)}) \leq \exp \left\{ \sup_{\eta \in (M_G^{2,0}(0,T))^d, |\eta| \leq C(\|f\|, \|f\|_{\text{lip}})} \mathbb{E}^G(\Phi(B^n) - H_T^G(\eta)) \right\} \quad (3.6)$$

Set $\phi(x_1, \dots, x_n) = e^{f(x_1, \dots, x_n)}$. Then $e^{-\|f\|} \leq \|\phi\| \leq e^{\|f\|}$, and there exists a constant $C_1(\|f\|, \|f\|_{\text{lip}}) \in (0, \infty)$ that is only dependent on $\|f\|$ and $\|f\|_{\text{lip}}$, such that

$$\|\phi\|_{\text{lip}} \leq e^{\|f\|} \sup_{y, z \in (\mathbb{R}^d)^n, y \neq z} \frac{e^{|f(y) - f(z)|} - 1}{|y - z|} \leq e^{\|f\|} \sup_{\lambda > 0} \frac{e^{\lambda \|f\|_{\text{lip}}} - 1}{\lambda} \leq C_1(\|f\|, \|f\|_{\text{lip}})$$

For $t_{n-1} \leq t \leq t_n$, set

$$\begin{aligned} v_n(t, x_1, \dots, x_{n-1}, x) &= \mathbb{E}^G(\phi(x_1, \dots, x_{n-1}, x + B_{t_n} - B_t)) \\ &= \mathbb{E}^G(\phi(x_1, \dots, x_{n-1}, x + \sqrt{t_n - t} B_1)), \end{aligned}$$

and for any $l = n-1, \dots, 1$, $t \in [t_{l-1}, t_l]$, define

$$\begin{aligned} v_l(t, x_1, \dots, x_{l-1}, x) &= \mathbb{E}^G(v_{l+1}(x_1, \dots, x_{l-1}, x + B_{t_l} - B_t, x + B_{t_l} - B_t)) \\ &= \mathbb{E}^G(v_{l+1}(x_1, \dots, x_{l-1}, x + \sqrt{t_l - t} B_1, x + \sqrt{t_l - t} B_1)). \end{aligned}$$

Then by the definition of G -Brownian motion, $v_l : [t_{l-1}, t_l] \times \mathbb{R}^d \rightarrow \mathbb{R}$, $l = 1, \dots, n$ are the solutions of equations:

$$\begin{cases} \frac{\partial}{\partial t} v_l(t, x_1, \dots, x_{l-1}, x) + G(D_x^2 v_l(t, x_1, \dots, x_{l-1}, x)) = 0, & t \in [t_{l-1}, t_l], \\ v_l(t_l, x_1, \dots, x_{l-1}, x) = v_{l+1}(t_l, x_1, \dots, x_{l-1}, x, x), \\ v_n(T, x_1, \dots, x_{n-1}, x) = \phi(x_1, \dots, x_{n-1}, x). \end{cases} \quad (3.7)$$

Since ϕ is bounded, by the regularity result of Krylov (Theorem 6.4.3 in [17]), and Section 4 in Appendix C of [22], there exists a constant $\alpha \in (0, 1)$ only depending on G , $\underline{\sigma}$, d and $\|\phi\|$ such that for each $l = 1, \dots, n$, and for any $\kappa \in (0, t_{l+1} - t_l)$,

$$\sup_{(x_1, \dots, x_l) \in (\mathbb{R}^d)^l} \|v_l(\cdot, x_1, \dots, x_{l-1}, \cdot)\|_{C^{1+\alpha/2, 2+\alpha}([t_l, t_{l+1}-\kappa] \times \mathbb{R}^d)} < \infty,$$

where for given real function u defined on $Q = [T_1, T_2] \times \mathbb{R}^d$, and given constants $\alpha, \beta \in (0, 1)$,

$$\begin{aligned} \|u\|_{C^{\alpha, \beta}(Q)} &= \sup_{\substack{x, y \in \mathbb{R}^d, x \neq y \\ s, t \in [T_1, T_2], s \neq t}} \frac{|u(s, x) - u(t, y)|}{|r - s|^\alpha + |x - y|^\beta}, \\ \|u\|_{C^{1+\alpha, 2+\beta}(Q)} &= \|u\|_{C^{\alpha, \beta}(Q)} + \|\partial_t u\|_{C^{\alpha, \beta}(Q)} + \sum_{i=1}^d \|\partial_{x^i} u\|_{C^{\alpha, \beta}(Q)} \\ &\quad + \sum_{i, j=1}^d \|\partial_{x^i x^j} u\|_{C^{\alpha, \beta}(Q)}. \end{aligned}$$

By the subadditivity of \mathbb{E}^G , for all $1 \leq l \leq n$, $(t, x_1, \dots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1}$,

$$\sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|v_l(t, x_1, \dots, x_{l-1}, x) - v_l(t, x_1, \dots, x_{l-1}, y)|}{|x - y|} \leq C_1(\|f\|, \|f\|_{lip}),$$

and for all $1 \leq l \leq n$, $(x_1, \dots, x_{l-1}, x) \in (\mathbb{R}^d)^l$,

$$|v_l(t, x_1, \dots, x_{l-1}, x) - v_l(s, x_1, \dots, x_{l-1}, x)| \leq \bar{\sigma} C_1(\|f\|, \|f\|_{lip}) |t - s|^{1/2}.$$

Therefore, $x \rightarrow \nabla_x v_l(t, x_1, \dots, x_{l-1}, x)$, $(t, x_1, \dots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1}$ are uniformly bounded.

Set $V_l(t, x_1, \dots, x_{l-1}, x) = \log v_l(t, x_1, \dots, x_{l-1}, x)$. Then for $(t, x) \in [t_{l-1}, t_l] \times \mathbb{R}^d$,

$$\frac{\partial V_l}{\partial t} = -G \left(\frac{D_x^2 v_l}{v_l} \right), \quad \nabla_x V_l = \frac{\nabla_x v_l}{v_l},$$

and

$$D_x^2 V_l = -(\partial_{x^i} V_l \partial_{x^j} V_l)_{i, j=1}^d + \frac{D_x^2 v_l}{v_l}.$$

Therefore, $\mathbb{R}^d \ni x \rightarrow \nabla_x V_l(t, x_1, \dots, x_{l-1}, x)$, $(t, x_1, \dots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1}$ are uniformly bounded, i.e., there exists a constant $C_2(\|f\|, \|f\|_{lip}) \in (0, \infty)$ that is only dependent on $\|f\|$ and $\|f\|_{lip}$, such that for all $1 \leq l \leq n$, $(t, x_1, \dots, x_{l-1}, x) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^l$,

$$|\nabla_x V_l(t, x_1, \dots, x_{l-1}, x)| \leq C_2(\|f\|, \|f\|_{lip})$$

and for any $\kappa \in (0, t_l - t_{l-1})$, $\mathbb{R}^d \ni x \rightarrow \nabla_x V_l(t, x_1, \dots, x_{l-1}, x)$, $(t, x_1, \dots, x_{l-1}) \in [t_{l-1}, t_l - \kappa] \times (\mathbb{R}^d)^{l-1}$ are uniformly Lipschitz continuous. Define

$$U_l(t, x_1, \dots, x_{l-1}, x) = \nabla_x V_l(t, x_1, \dots, x_{l-1}, x) I_{[t_{l-1}, t_l]}(t).$$

By the Picad iterative approach (cf. [14]), for any $\kappa \in (0, t_1)$, the stochastic differential equation

$$\begin{cases} dX_t^{(1)} = U_1(t, X_t^{(1)}) d\langle B \rangle_t + dB_t, & t \in [0, t_1 - \kappa], \\ X_0^{(1)} = 0, \end{cases}$$

has a unique continuous solution $\{X_t^{(1)}, t \in [0, t_1 - \kappa]\}$. From the arbitrariness of κ , and noting that for any $p \geq 1$,

$$\mathbb{E}^G \left(|X_t^{(1)} - X_s^{(1)}|^p \right) \leq 2^p \left(\|U_1\|^p \bar{\sigma}^p |t - s|^p + \bar{\sigma}^{p/2} |t - s|^{p/2} \right)$$

there exists a unique continuous process $\{X_t^{(1)}, t \in [0, t_1]\}$ such that

$$\begin{cases} dX_t^{(1)} = U_1(t, X_t^{(1)}) d\langle B \rangle_t + dB_t, & t \in [0, t_1], \\ X_0^{(1)} = 0, \end{cases}$$

Recursively, for any $1 \leq l \leq n$, there exists a unique continuous process $\{X_t^{(l)}, t \in [t_{l-1}, t_l]\}$, such that

$$\begin{cases} dX_t^{(l)} = U_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}) d\langle B \rangle_t + dB_t, & t \in [t_{l-1}, t_l], \\ X_{t_{l-1}}^{(l)} = X_{t_{l-1}}^{(l-1)}. \end{cases} \quad (3.8)$$

Define

$$\tilde{\eta}_t = U_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}), \quad t \in [t_{l-1}, t_l], \quad l = 1, \dots, n; \quad \tilde{\eta}_T = 0$$

and for any $l = 1, \dots, n$, and $t \in [t_{l-1}, t_l]$,

$$\begin{aligned} K_t^{(l)} = & \int_{t_{l-1}}^t \left(G \left(\frac{D_x^2 v_l}{v_l} \right) (t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)}) ds \right. \\ & \left. - \frac{1}{2} \frac{D_x^2 v_l}{v_l} (t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)}) d\langle B \rangle_s \right). \end{aligned}$$

Then, by Proposition 1.4 in [22], $\mathbb{E}^G(-K_t^{(l)}) = 0$ for any $t \in [t_{l-1}, t_l]$, and by Lemma 3.1, for any $t \in [t_{l-1}, t_l]$, $K_t^{(l)}(\omega) \geq 0$ and $t \rightarrow K_t^{(l)}$ is increasing for q.s. ω . Set $K_{t_l}^{(l)} = \lim_{t \uparrow t_l} K_t^{(l)}$.

By Itô formula for G -Brownian motion (cf. [14]), for any $t \in [t_{l-1}, t_l]$,

$$\begin{aligned} & V_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}) - V_l(t_{l-1}, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_{t_{l-1}}^{(l)}) \\ &= \int_{t_{l-1}}^t \frac{\partial V_l}{\partial t}(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)}) ds + \int_{t_{l-1}}^t \nabla_x V_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)}) dX_s^{(l)} \\ & \quad + \frac{1}{2} \int_{t_{l-1}}^t D_x^2 V_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)}) d\langle B \rangle_s \\ &= -K_t^{(l)} + H_t^G(\tilde{\eta}) - H_{t_{l-1}}^G(\tilde{\eta}) + \int_{t_{l-1}}^t \tilde{\eta}_s dB_s. \end{aligned}$$

Since $\mathbb{R}^d \ni x \rightarrow V_l(t, x_1, \dots, x_{l-1}, x)$, $(t, x_1, \dots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1}$ are uniformly Lipschitz continuous, and $[t_{l-1}, t_l] \ni t \rightarrow V_l(t, x_1, \dots, x_{l-1}, x)$, $(x_1, \dots, x_{l-1}, x) \in (\mathbb{R}^d)^l$ are 1/2-uniformly Hölder continuous, we have that

$$\lim_{t \uparrow t_l} \mathbb{E}^G \left(|V_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}) - V_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_{t_l}^{(l)})| \right) = 0.$$

Hence, $K_{t_l} \in L_G^1(\Omega_T)$, and $\lim_{t \uparrow t_l} \mathbb{E}^G \left(|K_t^{(l)} - K_{t_l}^{(l)}| \right) = 0$, $\mathbb{E}^G \left(-K_{t_l}^{(l)} \right) = 0$, and

$$\begin{aligned} & V_l(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}, X_{t_l}^{(l)}) - H_{t_l}^G(\tilde{\eta}) \\ &= V_{l-1}(t, X_{t_1}^{(1)}, \dots, X_{t_{l-1}}^{(l-1)}) - H_{t_{l-1}}^G(\tilde{\eta}) - K_{t_l}^{(l)} + \int_{t_{l-1}}^t \tilde{\eta}_s dB_s, \end{aligned}$$

which yields that

$$\begin{aligned} \Phi(B^{\tilde{\eta}}) - H_T^G(\tilde{\eta}) &= V_n(t, X_{t_1}^{(n)}, \dots, X_{t_n}^{(n)}) - H_{t_n}^G(\tilde{\eta}) \\ &= V_1(0, 0) - \sum_{l=1}^n K_{t_l}^{(l)} + \int_0^T \tilde{\eta}_s dB_s. \end{aligned}$$

Therefore,

$$\Phi(B^{\tilde{\eta}}) - H_T^G(\tilde{\eta}) - \int_0^T \tilde{\eta}_s dB_s = V_1(0, 0) - \sum_{l=1}^n K_{t_l}^{(l)}$$

and

$$\mathbb{E}^G \left(\Phi(B^{\tilde{\eta}}) - H_T^G(\tilde{\eta}) \right) = V_1(0, 0).$$

Since $\sum_{l=1}^n K_{t_l}^{(l)} \geq 0$, $q.s.$, we obtain that

$$\begin{aligned} \mathbb{E}^G \left(e^{\Phi(B)} \right) &= \mathbb{E}^G \left(\exp \left\{ \Phi(B^{\tilde{\eta}}) - \int_0^T \tilde{\eta}_s dB_s - H_T^G(\tilde{\eta}) \right\} \right) \\ &= e^{V_1(0,0)} \mathbb{E}^G \left(e^{-\sum_{l=1}^n K_{t_l}^{(l)}} \right) \\ &\leq \exp \left\{ \mathbb{E}^G \left(\Phi(B^{\tilde{\eta}}) - H_T^G(\tilde{\eta}) \right) \right\}. \end{aligned}$$

Now, for $m, N \geq 2/\min_{1 \leq i \leq n}(t_l - t_{l-1})$, define

$$\eta_s^{(m,N)} = \sum_{l=1}^n \sum_{k=1}^N \tilde{\eta}_{t_{l-1} + (k-1)(t_l - t_{l-1} - 1/m)/N} I_{[t_{l-1} + \frac{(k-1)(t_l - t_{l-1} - 1/m)}{N}, t_{l-1} + \frac{k(t_l - t_{l-1} - 1/m)}{N})}(s).$$

Then $\eta_s^{(m,N)} \in M_G^{2,0}(0, T)$, $|\eta_s^{(m,N)}| \leq C_2(\|f\|, \|f\|_{lip})$, and

$$\lim_{m \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}^G \left(\int_0^T |\eta_s^{(m,N)} - \tilde{\eta}_s|^2 \right) = 0.$$

Therefore,

$$\mathbb{E}^G \left(\Phi(B^{\tilde{\eta}}) - H_T^G(\tilde{\eta}) \right) \leq \sup_{\eta \in (M_G^{2,0}(0,T))^d, |\eta| \leq C_2(\|f\|, \|f\|_{lip})} \mathbb{E}^G \left(\Phi(B^\eta) - H_T^G(\eta) \right), \quad (3.9)$$

and so (3.6) holds.

For general bounded function $\Phi \in L_G^1(\Omega_T)$, choose a sequence $\{\Phi_n, n \geq 1\} \subset L_{ip}(\Omega_T)$ of uniformly bounded and Lipschitz continuous functions such that

$$\lim_{n \rightarrow \infty} \mathbb{E}^G(|\Phi_n - \Phi|^2) = 0.$$

Then

$$\lim_{n \rightarrow \infty} \mathbb{E}^G(|e^{\Phi_n} - e^\Phi|) = 0.$$

By the above proof, there exists a sequence of positive constants C_n such that

$$\log \mathbb{E}^G(\exp\{\Phi_n\}) \leq \sup_{\eta \in (M_G^{2,0}(0,T))^d, |\eta| \leq C_n} \mathbb{E}^G\left(\Phi_n(B^\eta) - H_T^G(\eta)\right).$$

Set $\mathbb{D} = \cup_{n \geq 1} \{\eta \in (M_G^{2,0}(0,T))^d; |\eta| \leq C_n\}$. Then

$$\log \mathbb{E}^G(\exp\{\Phi_n\}) \leq \sup_{\eta \in \mathbb{D}} \mathbb{E}^G\left(\Phi_n(B^\eta) - H_T^G(\eta)\right).$$

Since

$$\begin{aligned} & \left| \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi_n(B^\eta) - H_T^G(\eta)) - \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi(B^\eta) - H_T^G(\eta)) \right| \\ & \leq \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(|\Phi_n(B^\eta) - \Phi(B^\eta)|), \end{aligned}$$

and the same proof as (3.3) yields that

$$\lim_{n \rightarrow \infty} \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(|\Phi_n(B^\eta) - \Phi(B^\eta)|) = 0,$$

we obtain

$$\mathbb{E}^G(e^\Phi) \leq \exp \left\{ \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi(B^\eta) - H_T^G(\eta)) \right\} \quad (3.10)$$

which with together (3.5) yields the conclusion of Theorem 3.1. \square

4. AN ABSTRACT LARGE DEVIATION PRINCIPLE FOR FUNCTIONALS OF G -BROWNIAN MOTION

In this section, we apply the variation representation to study the large deviations for functionals of G -Brownian motion. The inverse of the Varadhan Lemma under a G -expectation is presented. An abstract large deviation principle for functionals of G -Brownian motion is obtained.

Let (\mathcal{Y}, ρ) be a Polish space and let $\Psi^\epsilon : \Omega_T \times \mathbb{A} \rightarrow \mathcal{Y}$ be a map. Set

$$Z^\epsilon := \Psi^\epsilon(\sqrt{\epsilon}B, \langle B \rangle).$$

Define

$$\mathbb{H} = \left\{ f(\cdot) = \int_0^\cdot f'(s)ds; f' \in L^2([0, T], \mathbb{R}^d) \right\}, \quad \|f\|_H = \|f'\|_{L^2}; \quad (4.1)$$

$$\mathbb{G} = \left\{ g(\cdot) = \int_0^\cdot g'(s)ds; g' \in L^2([0, T], \mathbb{R}^{d \times d}) \right\}, \quad \|g\|_G = \int_0^T \|g'(t)\|_{HS} dt; \quad (4.2)$$

and

$$\mathbb{A} = \left\{ g \in \mathbb{G}; g'(s) \in \Sigma \text{ for any } s \in [0, T] \right\}. \quad (4.3)$$

Then $(\mathbb{A}, \|\cdot\|_G)$ is a closed convex subset of \mathbb{G} . We also denote

$$\mathbb{H}_s = \{f \in \mathbb{H}; f'(t) = \theta_1 I_{[0,t_1]}(t) + \sum_{i=2}^m \theta_i I_{(t_{i-1}, t_i]}(t), 0 < t_1 < \dots < t_m = T, \theta_i \in \mathbb{R}^d\}$$

and

$$\|g\| := \sup_{t \in [0, T]} \|g(t)\|_{HS}, \quad g \in \mathbb{A}; \quad \|f\| := \sup_{t \in [0, T]} |f(t)|, \quad f \in \mathbb{H},$$

where $\|A\|_{HS} := \sqrt{\sum_{ij} a_{ij}^2}$ is the Hilbert-Schmidt norm of a matrix $A = (a_{ij})$. Define

$$\rho_{HG}((f_1, g_1), (f_2, g_2)) = \|f_1 - f_2\| + \|g_1 - g_2\|, \quad (f_1, g_1), (f_2, g_2) \in \mathbb{H} \times \mathbb{A}.$$

We introduce the following *Assumption (A)*:

(A0). For any $\Phi \in C_b(\mathcal{Y})$, $\Phi(Z^\epsilon)$ is quasi-continuous;

There exists a map $\Psi : \mathbb{H} \times \mathbb{A} \rightarrow \mathcal{Y}$ such that the following conditions (A1), (A2) and (A3) hold:

(A1). For each $N \geq 1$, if $f_n, n \geq 1, f \in \mathbb{H}, g_n \in \mathbb{A}$ and $g \in \mathbb{A}$ satisfy that $\|f_n\|_H \leq N, \|f\|_H \leq N, \|f_n - f\|_H \rightarrow 0$ and $\|g_n - g\|_G \rightarrow 0$, then

$$\Psi(f_n, g_n) \rightarrow \Psi(f, g);$$

(A2). For $\Phi \in C_b(\mathcal{Y})$, for each $r > 0$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{\substack{\eta \in (M_G^2(0, T))^d \cap \mathcal{B}_b(\Omega_T) \\ \int_0^T |\eta_s|^2 ds \leq r}} \mathbb{E}^G \left(\left| \Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right. \right. \\ \left. \left. - \Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right) = 0; \end{aligned}$$

(A3). There exists a sequence of continuous maps $\Psi^{(N)} : (\mathbb{H} \times \mathbb{A}, \rho_{HG}) \rightarrow (\mathcal{Y}, \rho)$ such that for each $l \in (0, \infty)$,

$$\lim_{N \rightarrow \infty} \sup_{\|f\|_H \leq l, g \in \mathbb{A}} \rho(\Psi(f, g), \Psi^{(N)}(f, g)) = 0.$$

Remark 4.1. *Assumption (A) is slightly different from the classical case (cf. [4]). We have an additional condition (A0). In the classical case, the condition (A0) is always true.*

Let $I : \mathcal{Y} \rightarrow [0, \infty]$ be defined by

$$I(y) = \inf_{(f, g) \in \mathbb{H} \times \mathbb{A}} \left\{ \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds; y = \Psi \left(\int_0^\cdot g'(s) f'(s) ds, g \right) \right\}. \quad (4.4)$$

Since $\varphi(t) = \int_0^t g'(s) f'(s) ds, t \in [0, T]$ yields that $f'(s) = g'(s)^{-1} \varphi'(s)$, it is easy to get the following representation of $I(y)$:

$$I(y) = \inf_{(f, g) \in \mathbb{H} \times \mathbb{A}} \{J(f, g), y = \Psi(f, g)\}, \quad y \in \mathcal{Y}, \quad (4.5)$$

where

$$J(f, g) = \begin{cases} \frac{1}{2} \int_0^T (f'(s), (g'(s))^{-1} f'(s)) ds, & (f, g) \in \mathbb{H} \times \mathbb{A}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.6)$$

Lemma 4.1. (1). Let $\Upsilon : (\mathbb{A}, \|\cdot\|) \rightarrow \mathbb{R}$ be a bounded continuous function. Then

$$\mathbb{E}^G(\Upsilon(\langle B \rangle)) = \sup_{g \in \mathbb{A}} \Upsilon(g). \quad (4.7)$$

(2). Let $\Phi \in C_b(\mathcal{Y})$, and let $\Psi : \mathbb{H} \times \mathbb{A} \rightarrow \mathcal{Y}$ satisfy (A1) and (A3). Then for each function $f \in \mathbb{H}$,

$$\begin{aligned} & \mathbb{E}^G \left(\Phi \circ \Psi \left(\int_0^\cdot f'(s) d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(f) \right) \\ &= \sup_{g \in \mathbb{A}} \left(\Phi \circ \Psi \left(\int_0^\cdot g'(s) f'(s) ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds \right). \end{aligned} \quad (4.8)$$

Proof. (1). Firstly, let us show (4.7) for

$$\Upsilon(g) = \psi(g_{t_1}, g_{t_2} - g_{t_1}, \dots, g_{t_m} - g_{t_{m-1}})$$

where ψ is bounded continuous in $(\mathbb{R}^{d \times d})^m$, and $0 < t_1 < t_2 < \dots < t_m \leq T$. Since $\langle B \rangle_t - \langle B \rangle_s$ is independent of Ω_s for any $s < t$, by Chapter III, Theorem 5.3 in [22], we have that

$$\begin{aligned} & \mathbb{E}^G(\psi(\langle B \rangle_{t_1}, \langle B \rangle_{t_2} - \langle B \rangle_{t_1}, \dots, \langle B \rangle_{t_m} - \langle B \rangle_{t_{m-1}})) \\ &= \sup_{\theta_1, \theta_2, \dots, \theta_m \in \Sigma} \psi(\theta_1 t_1, \theta_2(t_2 - t_1), \dots, \theta_m(t_m - t_{m-1})). \end{aligned}$$

For any $\theta_1, \theta_2, \dots, \theta_m \in \Sigma$, set $g'(t) = \theta_1 I_{[0, t_1]} + \sum_{i=2}^m \theta_i I_{(t_{i-1}, t_i]}$. Then $\psi(\theta_1 t_1, \theta_2(t_2 - t_1), \dots, \theta_m(t_m - t_{m-1})) = \psi(g(t_1), g(t_2) - g(t_1), \dots, g(t_m) - g(t_{m-1}))$. Therefore,

$$\begin{aligned} & \sup_{\theta_1, \theta_2, \dots, \theta_m \in \Sigma} \psi(\theta_1 t_1, \theta_2(t_2 - t_1), \dots, \theta_m(t_m - t_{m-1})) \\ & \leq \sup_{g \in \mathbb{A}} \psi(g(t_1), g(t_2) - g(t_1), \dots, g(t_m) - g(t_{m-1})). \end{aligned}$$

On the other hand, for any $g \in \mathbb{A}$, set $\theta_1 = g(t_1)/t_1$, and $\theta_i = (g(t_i) - g(t_{i-1}))/t_i$ for $i = 2, \dots, m$. Then $\theta_i \in \Sigma$ and $g(t_i) - g(t_{i-1}) = \theta_i(t_i - t_{i-1})$ for any $i = 1, \dots, m$, where $t_0 = 0$. Therefore,

$$\begin{aligned} & \sup_{\theta_1, \theta_2, \dots, \theta_m \in \Sigma} \psi(\theta_1 t_1, \theta_2(t_2 - t_1), \dots, \theta_m(t_m - t_{m-1})) \\ & \geq \sup_{g \in \mathbb{A}} \psi(g(t_1), g(t_2) - g(t_1), \dots, g(t_m) - g(t_{m-1})). \end{aligned}$$

Therefore, (4.7) holds in this case.

Next, we assume that Υ is Lipschitz continuous with respect to the uniform topology, i.e., there exists a constant $l > 0$ such that

$$|\Upsilon(g) - \Upsilon(f)| \leq l \sup_{t \in [0, T]} \|g(t) - f(t)\|_{HS} \quad \text{for all } g, f \in \mathbb{A}.$$

For each $N \geq 1$, set $t_i^N = \frac{iT}{N}$, $1 \leq i \leq N$. For any $(x_1, x_2, \dots, x_N) \in \Sigma^N$, set

$$x^{(N)}(t) := x_1(t \wedge t_1) + \sum_{i=2}^N x_i(t \wedge t_i - t \wedge t_{i-1}), \quad t \in [0, T],$$

and $\tilde{\psi}(x_1, x_2, \dots, x_N) = \Upsilon(x^{(N)})$. We can extend continuously $\tilde{\psi}(x_1, x_2, \dots, x_N)$ to $(\mathbb{R}^{d \times d})^N$. Define

$$\psi(x_1, x_2, \dots, x_N) := \tilde{\psi}\left(\frac{x_1}{t_1}, \frac{x_2}{t_2 - t_1}, \dots, \frac{x_N}{t_N - t_{N-1}}\right), \quad (x_1, x_2, \dots, x_N) \in (\mathbb{R}^{d \times d})^N.$$

For $g \in \mathbb{A}$, set

$$g^{(N)}(t) := \frac{g(t_1)}{t_1}(t \wedge t_1) + \sum_{i=2}^N \frac{(g(t_i) - g(t_{i-1}))}{t_i - t_{i-1}}(t \wedge t_i - t \wedge t_{i-1}), \quad t \in [0, T],$$

and define

$$\Upsilon^{(N)}(g) := \Upsilon(g^{(N)}) = \psi(g(t_1), g(t_2) - g(t_1), \dots, g(t_N) - g(t_{N-1})).$$

Then

$$\mathbb{E}^G(\Upsilon^{(N)}(\langle B \rangle)) = \sup_{g \in \mathbb{A}} \Upsilon^{(N)}(g).$$

and

$$|\Upsilon(g) - \Upsilon^{(N)}(g)| \leq 2l \max_{1 \leq i \leq N} \sup_{t \in [t_{i-1}, t_i]} \|g(t) - g(t_{i-1})\|_{HS} \leq 2l\bar{\sigma}T/N.$$

Therefore,

$$\mathbb{E}^G(\Upsilon(\langle B \rangle)) = \lim_{N \rightarrow \infty} \mathbb{E}^G(\Upsilon^{(N)}(\langle B \rangle)) = \lim_{N \rightarrow \infty} \sup_{g \in \mathbb{A}} \Upsilon^{(N)}(g) = \sup_{g \in \mathbb{A}} \Upsilon(g).$$

Now, by the proof of Lemma 3.1, Chapter VI in [22], for general bounded continuous Υ , we can choose a sequence of Lipschitz functions Υ_N such that $\Upsilon_N \uparrow \Upsilon$. Therefore

$$\begin{aligned} \mathbb{E}^G(\Upsilon(\langle B \rangle)) &= \sup_{\theta \in \mathcal{A}_{0, \infty}^\Gamma} \sup_{N \geq 1} E_{P_\theta}(\Upsilon_N(\langle B \rangle)) \\ &= \sup_{N \geq 1} \mathbb{E}^G(\Upsilon_N(\langle B \rangle)) = \sup_{N \geq 1} \sup_{g \in \mathbb{A}} \Upsilon_N(g) = \sup_{g \in \mathbb{A}} \Upsilon(g). \end{aligned}$$

(2). Choose a sequence of simple functions $f'_N = \theta_1^N I_{[0, t_1^N]} + \sum_{i=2}^{m_N} \theta_i^N I_{(t_{i-1}^N, t_i^N]}$ such that $\sup_{N \geq 1} \|f_N\|_H < \infty$ and $\int_0^T |f'(s) - f'_N(s)|^2 ds \rightarrow 0$ as $N \rightarrow \infty$. Then

$$\sup_{g \in \mathbb{A}} \left| \int_0^T (f'(s), g'(s) f'(s)) ds - \int_0^T (f'_N(s), g'(s) f'_N(s)) ds \right| \rightarrow 0,$$

and

$$\sup_{g \in \mathbb{A}} \sup_{t \in [0, T]} \left| \int_0^t g'(s) f'(s) ds - \int_0^t g'(s) f'_N(s) ds \right| \rightarrow 0.$$

Let $\Psi^{(N)} : (\mathbb{H} \times \mathbb{A}, \rho_{HG}) \rightarrow (\mathcal{Y}, \rho)$ such that for any $l \in (0, \infty)$,

$$\lim_{N \rightarrow \infty} \sup_{\|\varphi\|_H \leq l, g \in \mathbb{A}} \rho(\Psi(\varphi, g), \Psi^{(N)}(\varphi, g)) = 0.$$

Define

$$\Phi^{(N)}(g) = \Phi \circ \Psi_N \left(\int_0^\cdot g'(s) f'_N(s) ds, g \right) - \frac{1}{2} \int_0^T (f'_N(s), g'(s) f'_N(s)) ds$$

Then

$$\lim_{N \rightarrow \infty} \sup_{g \in \mathbb{A}} \left| \Phi^{(N)}(g) - \left(\Phi \circ \Psi \left(\int_0^\cdot g'(s) f'(s) ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds \right) \right| = 0,$$

and by (1),

$$\mathbb{E}^G(\Phi^{(N)}(\langle B \rangle)) = \sup_{g \in \mathbb{A}} \Phi^{(N)}(g),$$

Therefore, (4.8) holds. \square

Lemma 4.2. *Let (A2) hold. Then for any $\Phi \in C_b(\mathcal{Y})$ and each $N \geq 1$,*

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{\eta \in (M_G^{2,0}(0, T))^d \cap \mathcal{B}_b(\Omega_T)} \mathbb{E}^G \left(\left| \Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right. \right. \\ \left. \left. - \Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right) = 0; \end{aligned}$$

Proof. For $\eta \in (M_G^{2,0}(0, T))^d \cap \mathcal{B}_b(\Omega_T)$, we can write $\eta_s = \sum_{k=1}^n \eta_{t_{k-1}} I_{[t_{k-1}, t_k)}(s)$. For $r \in (0, \infty)$ fixed, for any $\delta > 0$, let $\phi(x) \in \text{lip}(\mathbb{R})$ satisfy $0 \leq \phi \leq 1$, $\phi(x) = 1$ for all $|x| \leq r$ and $\phi(x) = 0$ for all $|x| \geq r + \delta$. Define

$$\hat{\eta}_t = \eta_t \phi \left(\int_0^t |\eta_s|^2 ds \right).$$

Then

$$\left| \int_0^T |\hat{\eta}_s|^2 ds \right| \leq r + \delta, \quad \left\{ \int_0^T |\eta_s|^2 ds \leq r \right\} \subset \{ \hat{\eta}_s = \eta_s \text{ for any } s \in [0, T] \}.$$

Set

$$\hat{\eta}_s^N := \sum_{k=1}^n \sum_{j=1}^N \hat{\eta}_{t_{k-1} + (j-1)(t_k - t_{k-1})/N} I_{[t_{k-1} + \frac{(j-1)(t_k - t_{k-1})}{N}, t_{k-1} + \frac{j(t_k - t_{k-1})}{N})}(s).$$

Then

$$\mathbb{E}^G \left(\int_0^T |\hat{\eta}_s^N - \hat{\eta}_s| ds \right) \leq \frac{\|\phi\|_{\text{lip}}}{2} \mathbb{E}^G \left(\sum_{k=1}^n \sum_{j=1}^N |\eta_{t_{k-1}}|^3 \frac{(t_k - t_{k-1})^2}{N^2} \right) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Therefore, for any $t \in [0, T]$,

$$\mathbb{E}^G \left(\left| \int_0^t \hat{\eta}_s^N d\langle B \rangle_s - \int_0^t \hat{\eta}_s d\langle B \rangle_s \right| \right) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

In particular, on $\left\{ \int_0^T |\eta_s|^2 ds \leq r \right\}$,

$$\int_0^\cdot \eta_s d\langle B \rangle_s = \int_0^\cdot \hat{\eta}_s d\langle B \rangle_s, \text{ q.s.},$$

and

$$\Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \hat{\eta}_s d\langle B \rangle_s, \langle B \rangle \right) = \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right), \text{ q.s.}$$

Therefore, for any $\Phi \in C_b(\mathcal{Y})$ and each $N \geq 1$, for all $\eta \in (M_G^{2,0}(0, T))^d \cap \mathcal{B}_b(\Omega_T)$ with $\int_0^T \mathbb{E}^G(|\eta_s|^2) ds \leq N$,

$$\begin{aligned} & \mathbb{E}^G \left(\left| \Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) - \Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right) \\ & \leq \frac{2\|\Phi\|N}{r} + \mathbb{E}^G \left(\left| \Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \tilde{\eta}_s d\langle B \rangle_s, \langle B \rangle \right) \right. \right. \\ & \quad \left. \left. - \Phi \circ \Psi \left(\int_0^\cdot \tilde{\eta}_s d\langle B \rangle_s, \langle B \rangle \right) \right| I_{\{\int_0^T |\eta_s|^2 ds \leq r\}} \right) \\ & \leq \frac{2\|\Phi\|N}{r} + \sup_{\substack{\eta \in (M_G^{2,0}(0, T))^d \cap \mathcal{B}_b(\Omega_T) \\ \int_0^T |\eta_s|^2 ds \leq r + \delta}} \mathbb{E}^G \left(\left| \Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right. \right. \\ & \quad \left. \left. - \Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right). \end{aligned}$$

First, letting $\epsilon \rightarrow 0$, then $r \rightarrow \infty$, by (A2), we obtain the conclusion of the lemma. \square

Theorem 4.1. *Suppose that the assumption (A) holds. Then*

- (1). *For any $L \in [0, \infty)$, $C_L := \{y; I(y) \leq L\}$ is compact in \mathcal{Y} ;*
- (2). *For any $\Phi \in C_b(\mathcal{Y})$,*

$$\lim_{\epsilon \rightarrow 0} \left| \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) - \sup_{y \in \mathcal{Y}} \{ \Phi(y) - I(y) \} \right| = 0. \quad (4.9)$$

Proof. (1). First, we prove that $C_L = \cap_{n \geq 1} \Gamma_{L+1/n}$, where

$$\Gamma_{L+1/n} = \left\{ \Psi(f, g) : J(f, g) \leq L + \frac{1}{n}, (f, g) \in \mathbb{H} \times \mathbb{A} \right\}.$$

In fact, for $y \in C_L$ given, for each $n \geq 1$, choose $f_n \in \mathbb{H}$, $g_n \in \mathbb{A}$ such that $y = \Psi(f_n, g_n)$ and $J(f_n, g_n) \leq L + \frac{1}{n}$. Since $n \geq 1$ is arbitrary, we have $C_L \subseteq \cap_{n \geq 1} \Gamma_{L+1/n}$. Conversely, suppose $y \in \cap_{n \geq 1} \Gamma_{L+1/n}$. Then, for some $f_n \in \mathbb{H}$, $g_n \in \mathbb{A}$

with $y = \Psi(f_n, g_n)$, we have that $J(f_n, g_n) \leq L + 1/n$. Therefore, $I(y) \leq L + \frac{1}{n}$. Letting $n \rightarrow \infty$, we obtain $I(y) \leq L$. Thus $y \in C_L$, and in turn, $\cap_{n \geq 1} \Gamma_{L+1/n} \subseteq \bar{C}_L$ follows.

(2). From Theorem 3.1, we have

$$\begin{aligned}
& \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{1}{\epsilon} \Phi(Z^\epsilon) \right\} \right) \\
&= \sup_{\eta \in (M_G^{2,0}(0,T))^d \cap \mathcal{B}_b(\Omega_T)} \mathbb{E}^G \left(\Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B^\eta, \langle B \rangle \right) - H_T^G(\sqrt{\epsilon} \eta) \right) \\
&= \sup_{\eta \in (M_G^{2,0}(0,T))^d \cap \mathcal{B}_b(\Omega_T)} \mathbb{E}^G \left(\Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right) \\
&= \sup_{\substack{\eta \in (M_G^{2,0}(0,T))^d \cap \mathcal{B}_b(\Omega_T) \\ \int_0^T \mathbb{E}^G(|\eta_s|^2) ds \leq \frac{4\|\Phi\|}{\underline{\sigma}}}} \mathbb{E}^G \left(\Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right)
\end{aligned}$$

where $\|\Phi\| = \sup_{y \in \mathcal{Y}} |\Phi(y)|$, and the last equality is due to that if $\int_0^T \mathbb{E}^G(|\eta_s|^2) ds > \frac{4\|\Phi\|}{\underline{\sigma}}$, then

$$\mathbb{E}^G \left(\Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right) \leq -\|\Phi\|.$$

Therefore, by (A2) and Lemma 4.2, as $\epsilon \rightarrow 0$,

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} \left| \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) \right. \\
& \quad \left. - \sup_{\substack{\eta \in (M_G^{2,0}(0,T))^d \cap \mathcal{B}_b(\Omega_T) \\ \int_0^T \mathbb{E}^G(|\eta_s|^2) ds \leq \frac{4\|\Phi\|}{\underline{\sigma}}}} \mathbb{E}^G \left(\Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right) \right| = 0. \quad (4.10)
\end{aligned}$$

Since for each $\eta \in (M_G^{2,0}(0,T))^d \cap \mathcal{B}_b(\Omega_T)$,

$$\begin{aligned}
& \Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \\
& \leq \sup_{(f,g) \in \mathbb{H} \times \mathbb{A}} \left(\Phi \circ \Psi \left(\int_0^\cdot g'(s) f'(s) ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds \right) \\
& = \sup_{y \in \mathcal{Y}} \sup_{(f,g) \in \mathbb{H} \times \mathbb{A}, y = \Psi(\int_0^\cdot g'(s) f'(s) ds, g)} \left(\Phi(y) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds \right) \\
& = \sup_{y \in \mathcal{Y}} \{ \Phi(y) - I(y) \}, \quad q.s.,
\end{aligned}$$

we obtain the upper bound:

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) \leq \sup_{y \in \mathcal{Y}} \{ \Phi(y) - I(y) \}.$$

From Theorem 3.1, we also have that

$$\begin{aligned} & \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{1}{\epsilon} \Phi(Z^\epsilon) \right\} \right) \\ & \geq \sup_{f \in \mathbb{H}_s; \|f\|_H \leq \frac{4\|\Phi\|}{\epsilon}} \mathbb{E}^G \left(\Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot f'_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(f) \right). \end{aligned}$$

Thus, by (A2), (A3) and Lemma 4.1,

$$\begin{aligned} & \liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) \\ & \geq \sup_{f \in \mathbb{H}_s, \|f\|_H \leq \frac{4\|\Phi\|}{\epsilon}} \mathbb{E}^G \left(\Phi \circ \Psi \left(\int_0^\cdot f'_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(f) \right) \\ & = \sup_{f \in \mathbb{H}_s, \|f\|_H \leq \frac{4\|\Phi\|}{\epsilon}} \sup_{g \in \mathbb{A}} \left(\Phi \circ \Psi \left(\int_0^\cdot g'(s) f'(s) ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds \right) \\ & = \sup_{\|f\|_H \leq \frac{4\|\Phi\|}{\epsilon}} \sup_{g \in \mathbb{A}} \left(\Phi \circ \Psi \left(\int_0^\cdot g'(s) f'(s) ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s)) ds \right). \end{aligned}$$

Then, letting $N \rightarrow \infty$, we obtain the lower bound:

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) \geq \sup_{y \in \mathcal{Y}} \{ \Phi(y) - I(y) \}.$$

Therefore, (4.9) is valid. \square

Theorem 4.2. *Suppose that the assumption (A) holds. Then $\{Z^\epsilon, \epsilon > 0\}$ satisfies the large deviation principle in \mathcal{Y} with the rate function $I(y)$, i.e., for any closed subset $F \subset \mathcal{Y}$,*

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log c^G(Z^\epsilon \in F) \leq - \inf_{y \in F} I(y), \quad (4.11)$$

and for any open subset $O \subset \mathcal{Y}$,

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log c^G(Z^\epsilon \in O) \geq - \inf_{y \in O} I(y). \quad (4.12)$$

Proof. This is a consequence of Theorem 4.1. Its proof is the same as probability measure case. For given open set O , for any $y \in O$, choose continuous map $\Phi : \mathcal{Y} \rightarrow [0, 1]$ such that $\Phi(y) = 1$ and for any $z \in O^c$, $\Phi(z) = 0$. For any $m \geq 1$, set $\Phi_m(z) := m(\Phi(z) - 1)$, $z \in \mathcal{Y}$. Then

$$\mathbb{E}^G \left(\exp \left\{ \frac{1}{\epsilon} \Phi_m(Z^\epsilon) \right\} \right) \leq e^{-\frac{m}{\epsilon}} c^G(Z^\epsilon \in O^c) + c^G(Z^\epsilon \in O).$$

Therefore,

$$\begin{aligned} & \max \left\{ \liminf_{\epsilon \rightarrow 0} \epsilon \log c^G(Z^\epsilon \in O), -m \right\} \\ & \geq \liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{1}{\epsilon} \Phi_m(Z^\epsilon) \right\} \right) = \sup_{z \in \mathcal{Y}} \{ \Phi(z) - I(z) \} \geq -I(y). \end{aligned}$$

Letting $m \rightarrow +\infty$, we obtain the lower bound.

Next, let us show the upper bound. For closed set F given, for any $y \notin F$, choose continuous map $\Phi_y : E \rightarrow [0, 1]$ such that $\Phi_y(y) = 1$ and for any $z \in F$, $\Phi_y(z) = 0$. For any finite set $A \subset F^c$, set $\Phi_A(z) = \max_{y \in A} \Phi_y(z)$. Then

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} \epsilon \log c^G(Z^\epsilon \in F) & \leq \inf_{\substack{A \subset F^c \\ \text{finite}}} \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{-m \Phi_A(Z^\epsilon)}{\epsilon} \right\} \right) \\ & = - \sup_{\substack{A \subset F^c \\ \text{finite}}} \inf_{z \in \mathcal{Y}} \{ m \Phi_A(z) + I(z) \}. \end{aligned}$$

Without loss of generality, we assume that $l := \sup_{\substack{A \subset F^c \\ \text{finite}}} \inf_{z \in \mathcal{Y}} J_A(z) < \infty$, where $J_A(z) = m \Phi_A(z) + I(z)$. Then $\{z; J_A(z) \leq l\}$ is nonempty compact set for any finite A . Therefore, $\cap_{A \subset F^c, \text{finite}} \{z; J_A(z) \leq l\}$ is nonempty, and so

$$l \geq \inf_{z \in \mathcal{Y}} \sup_{\substack{A \subset F^c \\ \text{finite}}} J_A(z) = \min \left\{ m + \inf_{z \in F^c} I(z), \inf_{z \in F} I(z) \right\} \xrightarrow{m \rightarrow \infty} \inf_{z \in F} I(z),$$

which yields (4.11). □

5. LARGE DEVIATIONS FOR STOCHASTIC FLOWS DRIVEN BY G -BROWNIAN MOTION

The homeomorphic property with respect to initial values of the solution for stochastic differential equations driven by G -Brownian motion was obtained in [14] and the large deviations for solutions $\{X^\epsilon(x, t), t \in [0, T], \} \subset C([0, T], \mathbb{R}^p)$ of small perturbation stochastic differential equations starting from x (fixed) by G -Brownian motion were studied in [15] by exponential estimates and discretization/approximation techniques. In this section, we consider large deviations for the flows $\{X^\epsilon(x, t), (x, t) \in \mathbb{R}^p \times [0, T]\} \subset C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$. The quasi continuity of the flows is proved. A Kolmogorov criterion on weak convergence under G -expectations is given. A large deviation principle for the flows is established under the Lipschitz condition. In the classical framework, Large deviations for stochastic flows have been studied extensively (see [1], [2], [6], [13], [18], [23] and references therein). For general theory of large deviations and random perturbations, we refer to [8], [11], and [12].

For positive number $p \geq 1$ given, for each $N \geq 1$, $\psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$, set

$$\|\psi\|_N = \sup_{x \in [-N, N]^p, t \in [0, T]} |\psi(x, t)|,$$

and define

$$\rho(\psi_1, \psi_2) = \sum_{N=1}^{\infty} \frac{1}{2^N} \min\{\|\psi_1 - \psi_2\|_N, 1\}, \quad \psi_1, \psi_2 \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p).$$

Then $(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p), \rho)$ is a separable metric space.

Consider the following small perturbation stochastic differential equation driven by a d -dimensional G -Brownian motion B :

$$X^\epsilon(x, t) = x + \int_0^t b^\epsilon(X^\epsilon(x, s))ds + \sqrt{\epsilon} \int_0^t \sigma^\epsilon(X^\epsilon(x, s))dB_s + \int_0^t h^\epsilon(X^\epsilon(x, s))d\langle B \rangle_s, \quad (5.1)$$

where

$$b^\epsilon : \mathbb{R}^p \rightarrow \mathbb{R}^p; \quad \sigma^\epsilon = (\sigma_{i,j}^\epsilon)_{1 \leq i \leq p, 1 \leq j \leq d} : \mathbb{R}^p \rightarrow \mathbb{R}^p \otimes \mathbb{R}^d,$$

and

$$h^\epsilon = (h^{\epsilon,k})_{1 \leq k \leq p} = ((h_{ij}^{\epsilon,k})_{1 \leq i,j \leq d})_{1 \leq k \leq p} : \mathbb{R}^p \mapsto (\mathbb{R}^{d \times d})^p, \quad \epsilon \geq 0$$

satisfy the following conditions:

(H1). b^ϵ , σ^ϵ and h^ϵ , $\epsilon \geq 0$ are uniformly Lipschitz continuous, i.e., there exists a constant $L > 0$ such that for any $x, y \in \mathbb{R}^p$,

$$\max \left\{ |b^\epsilon(x) - b^\epsilon(y)|, \|\sigma^\epsilon(x) - \sigma^\epsilon(y)\|_{HS}, \max_{1 \leq k \leq p} \|h^{\epsilon,k}(x) - h^{\epsilon,k}(y)\|_{HS} \right\} \leq L|x - y|.$$

(H2). b^ϵ , σ^ϵ and h^ϵ converge uniformly to $b := b^0$, $\sigma := \sigma^0$ and $h := h^0$ respectively, i.e.,

$$\limsup_{\epsilon \rightarrow 0} \max_{x \in \mathbb{R}^p} \left\{ |b^\epsilon(x) - b(x)|, \|\sigma^\epsilon(x) - \sigma(x)\|_{HS}, \max_{1 \leq k \leq p} \|h^{\epsilon,k}(x) - h^k(x)\|_{HS} \right\} = 0.$$

Then by Theorem 4.1 in [14] and the Kolmogorov criterion under G -expectation (cf. Theorem 1.36, Chapter VI in [22]), the SDE (5.1) has a unique solution $X^\epsilon = \{X^\epsilon(x, t), x \in \mathbb{R}^p, t \in [0, T]\} \subset C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$ and $X^\epsilon(x, t) \in L_G^2(\Omega_T)$ for all $(x, t) \in \mathbb{R}^p \times [0, T]$. Furthermore, there exists a map $\Psi^\epsilon : \Omega_T \times \mathbb{A} \rightarrow C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$ such that

$$\Psi^\epsilon(\sqrt{\epsilon}B, \langle B \rangle) = X^\epsilon. \quad (5.2)$$

For any $(f, g) \in \mathbb{H} \times \mathbb{A}$, let $\Psi(f, g)(x, t) \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$ be a unique solution of the following ordinary differential equation:

$$\begin{aligned} \Psi(f, g)(x, t) = & x + \int_0^t b(\Psi(f, g)(x, s))ds + \int_0^t \sigma(\Psi(f, g)(x, s))f'(s)ds \\ & + \int_0^t h(\Psi(f, g)(x, s))dg(s). \end{aligned} \quad (5.3)$$

Theorem 5.1. *Let (H1) and (H2) hold. Let $X^\epsilon = \{X^\epsilon(x, t), x \in \mathbb{R}^p, t \in [0, T]\}$ be a unique solution of the SDE (5.1). Then*

(1). *For any $\Phi \in C_b(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p))$,*

$$\lim_{\epsilon \rightarrow 0} \left| \epsilon \log \mathbb{E}^G \left(\exp \left\{ \frac{\Phi(X^\epsilon)}{\epsilon} \right\} \right) - \sup_{\psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)} \{\Phi(\psi) - I(\psi)\} \right| = 0, \quad (5.4)$$

where

$$I(\psi) = \inf_{(f, g) \in \mathbb{H} \times \mathbb{A}} \{J(f, g), \psi = \Psi(f, g)\}, \quad \psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p). \quad (5.5)$$

(2). *For any closed subset F in $(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p), \rho)$,*

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log c^G(X^\epsilon \in F) \leq - \inf_{\psi \in F} I(\psi) \quad (5.6)$$

and for any open subset O in $(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p), \rho)$,

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log c^G(X^\epsilon \in O) \geq - \inf_{\psi \in O} I(\psi), \quad (5.7)$$

Proof. By Theorem 4.2, we only need to verify the conditions (A0), (A1), (A2) and (A3) for $\mathcal{Y} = C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$, $Z^\epsilon = X^\epsilon$ and Ψ defined by (5.3). These will be given in Lemma 5.1, Lemma 5.2 and Lemma 5.3. \square

Remark 5.1. *In particular, Theorem 5.1 yields that $\{\{\sqrt{\epsilon}B_t, t \in [0, T]\}, \epsilon > 0\}$ satisfies a large deviation principle, which was first obtained in [15] by the subadditive method.*

Lemma 5.1. *Assume that (H1) and (H2) hold. Let $X = \{X(x, t), x \in \mathbb{R}^p, t \in [0, T]\}$ be a unique solution of the SDE:*

$$X(x, t) = x + \int_0^t b(X(x, s))ds + \int_0^t \sigma(X(x, s))dB_s + \int_0^t h(X(x, s))d\langle B \rangle_s. \quad (5.8)$$

Then for any $\Phi \in C_b(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p))$, $\Phi(X)$ is quasi-continuous.

Proof. First, we assume that $\Phi \in C_b(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p))$ is Lipschitz continuous, i.e., there exists a constant $l > 0$ such that

$$|\Phi(\psi) - \Phi(\varphi)| \leq l\rho(\psi, \varphi) \quad \text{for all } \psi, \varphi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p).$$

For any $N \geq 1$, for each $\psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$, set $\psi^{(N)}(x, t) = \psi((-N) \vee x \wedge N, t)$, where $(-N) \vee x \wedge N = ((-N) \vee x_1 \wedge N, \dots, (-N) \vee x_p \wedge N)$. For given $N \geq 1$, for each $\psi = (\psi_1, \dots, \psi_p) \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$, for any $(x_1, \dots, x_p, x_{p+1}) \in [0, 1]^{p+1}$, set

$$\tilde{\psi}_j(x_1, \dots, x_p, x_{p+1}) = \psi_j^{(N)}(N(2x_1 - 1), \dots, N(2x_p - 1), Tx_{p+1}).$$

For any $m \geq 1$, the Bernstein polynomial of $\tilde{\psi}_j$ is defined by

$$B_m(\tilde{\psi}_j)(x_1, \dots, x_p, x_{p+1}) = \sum_{1 \leq i_1, \dots, i_{p+1} \leq m} \tilde{\psi}_j\left(\frac{i_1}{m}, \dots, \frac{i_p}{m}, \frac{i_{p+1}}{m}\right) \prod_{k=1}^{p+1} \binom{m}{i_k} x_k^{i_k} (1-x_k)^{m-i_k}.$$

Then, by Bernstein's theorem (cf. Theorem 3.1 and its proof in [16]), we have that

$$\begin{aligned} & \left| \tilde{\psi}_j(x_1, \dots, x_p, x_{p+1}) - B_m(\tilde{\psi}_j)(x_1, \dots, x_p, x_{p+1}) \right| \\ & \leq \sup_{\sum_{k=1}^{p+1} |x_k - y_k|^2 \leq 1/m} \left| \tilde{\psi}_j(x_1, \dots, x_p, x_{p+1}) - \tilde{\psi}_j(y_1, \dots, y_p, y_{p+1}) \right| \\ & \quad + \frac{p+1}{2m} \sup_{(x_1, \dots, x_p, x_{p+1}) \in [0,1]^{p+1}} \left| \tilde{\psi}_j(x_1, \dots, x_p, x_{p+1}) \right|. \end{aligned}$$

Since $X(x, t) \in L_G^2(\Omega_T)$ for all $(x, t) \in \mathbb{R}^p \times [0, T]$, we have that

$$\tilde{X}_j \left(\frac{i_1}{m}, \dots, \frac{i_p}{m}, \frac{i_{p+1}}{m} \right) \in L_G^2(\Omega_T), j = 1, \dots, p, 1 \leq i_1, \dots, i_{p+1} \leq m;$$

and

$$B_m(\tilde{X}_j)(x_1, \dots, x_p, x_{p+1}) \in L_G^2(\Omega_T) \text{ for all } (x_1, \dots, x_p, x_{p+1}) \in [0, 1]^{p+1}.$$

For $x \in \mathbb{R}^p, t \in [0, T]$, Set

$$\begin{aligned} & X^{N,m}(x, t) \\ & = \left(B_m(\tilde{X}_1) \left((-1) \vee \frac{x+N}{2N} \wedge 1, \frac{t}{T} \right), \dots, B_m(\tilde{X}_p) \left((-1) \vee \frac{x+N}{2N} \wedge 1, \frac{t}{T} \right) \right). \end{aligned}$$

Noting that $\Phi(X^{N,m})$ is a continuous function of $\tilde{X}_j \left(\frac{i_1}{m}, \dots, \frac{i_p}{m}, \frac{i_{p+1}}{m} \right), j = 1, \dots, p, 1 \leq i_1, \dots, i_{p+1} \leq m$, we obtain $\Phi(X^{N,m}) \in L_G^2(\Omega_T)$.

By Theorem 4.1 in [14], for any $q \geq 2$,

$$\mathbb{E}^G(|X(x, t) - X(y, s)|^q) \leq C_{q,T}(|x - y|^q + |s - t|^{q/2}). \quad (5.9)$$

This yields by the Kolmogorov criterion under G -expectation (cf. Theorem 1.36, Chapter VI in [22])) that for each $1 \leq j \leq p$,

$$\lim_{m \rightarrow \infty} \mathbb{E}^G \left(\sup_{\sum_{k=1}^{p+1} |x_k - y_k|^2 \leq 1/m} \left| \tilde{X}_j(x_1, \dots, x_p, x_{p+1}) - \tilde{X}_j(y_1, \dots, y_p, y_{p+1}) \right|^2 \right) = 0,$$

and

$$\mathbb{E}^G \left(\sup_{(x_1, \dots, x_p, x_{p+1}) \in [0,1]^{p+1}} \left| \tilde{X}_j(x_1, \dots, x_p, x_{p+1}) \right|^2 \right) < \infty.$$

Therefore,

$$\lim_{m \rightarrow \infty} \mathbb{E}^G \left(\sup_{x \in [-N, N]^p, t \in [0, T]} |X(x, t) - X^{N,m}(x, t)|^2 \right) = 0,$$

and by

$$|\Phi(X) - \Phi(X^{N,m})| \leq l \sup_{x \in [-N, N]^p, t \in [0, T]} |X(x, t) - X^{N,m}(x, t)| + \frac{l}{2^{N-1}},$$

we obtain

$$\lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} \overline{\mathbb{E}}^G \left(|\Phi(X) - \Phi(X^{N,m})|^2 \right) = 0,$$

which implies that $\Phi(X) \in L_G^1(\Omega_T)$, and so $\Phi(X)$ is quasi-continuous.

For general $\Phi \in C_b(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p))$, set $M = \sup_{\psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)} |\Phi(\psi)|$. For any $N \geq 1$, set

$$\Phi^{(N)}(\psi) = \inf_{\varphi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)} \{ \Phi(\varphi) + N \|\psi - \varphi\|_N \}, \quad \psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p).$$

Then (cf. Lemma 3.1, Chapter VI in [22]), $|\Phi^{(N)}| \leq M$,

$$|\Phi^{(N)}(\psi) - \Phi^{(N)}(\varphi)| \leq N \|\psi - \varphi\|_N, \quad \psi, \varphi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p),$$

and for any $\psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$, $\Phi^{(N)}(\psi) \uparrow \Phi(\psi)$ as $N \rightarrow \infty$. Therefore, $\Phi^{(N)}(X)$ is quasi-continuous for all $N \geq 1$. For any $\delta > 0$, choose a compact subset $K \subset \Omega_T$ such that $c^G(K^c) < \delta$ and for all $N \geq 1$, $\Phi^{(N)}(X)$ is continuous on K . By Dini's Theorem, $\Phi^{(N)}(X)$ converges uniformly to $\Phi(X)$ on K , and so $\Phi(X)$ is continuous on K . Thus, $\Phi(X)$ is quasi-continuous. \square

Lemma 5.2. *Assume that (H1) and (H2) hold.*

(1). *For each $N \geq 1$, if $f_n, n \geq 1$, $f \in \mathbb{H}$, $g_n \in \mathbb{A}$ and $g \in \mathbb{A}$ satisfy that $\|f_n\|_H \leq N$, $\|f\|_H \leq N$, $\|f_n - f\|_H \rightarrow 0$ and $\|g_n - g\|_G \rightarrow 0$, then*

$$\Psi(f_n, g_n) \rightarrow \Psi(f, g).$$

(2). *For $\Phi \in C_b(C(\mathbb{R}^p \times [0, T], \mathbb{R}^p))$, for each $N \geq 1$,*

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{\eta \in (M_G^2(0, T))^{d \cap \mathcal{B}_b(\Omega_T)} \int_0^T |\eta_s|^2 ds \leq N} \mathbb{E}^G \left(\left| \Phi \circ \Psi^\epsilon \left(\sqrt{\epsilon} B. + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right. \right. \\ \left. \left. - \Phi \circ \Psi \left(\int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right) = 0. \end{aligned} \quad (5.10)$$

Proof. (1). For any $m \geq 1$, set

$$M_m = \sup_{|x| \leq m, t \in [0, T]} (\|\sigma(\Psi(f, g)(x, s))\|_{HS} + \max_{1 \leq k \leq p} \|h^k(\Psi(f, g)(x, s))\|_{HS}).$$

Then, there exists a constant $M \in (0, \infty)$ such that, on $\{|x| \leq m, t \in [0, T]\}$,

$$\begin{aligned} & |\Psi(f, g)(x, t) - \Psi(f_n, g_n)(x, t)| \\ & \leq M \int_0^t |\Psi(f, g)(x, s) - \Psi(f_n, g)(x, s)| (1 + |f'_n(s)| + \|g'_n(s)\|_{HS}) ds \\ & \quad + M_m \int_0^t (|f'_n(s) - f'(s)| + \|g'_n(s) - g'(s)\|_{HS}) ds. \end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
& \sup_{|x| \leq m, t \in [0, T]} |\Psi(f, g)(x, t) - \Psi(f_n, g)(x, t)| \\
& \leq M_m \int_0^T (|f'_n(s) - f'(s)| + \|g'_n(s) - g'(s)\|_{HS}) ds e^{M \int_0^T (1 + |f'_n(t)| + \|g'_n(t)\|_{HS}) dt} \\
& \leq M_m \left(\sqrt{T} \|f_n - f\|_H + \|g_n - g\|_G \right) e^{M(T + \sqrt{NT} + p\bar{\sigma}T)} \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

(2). For any $\eta \in (M_G^2(0, T))^d$ with $\int_0^T |\eta_s|^2 ds \leq r$, set $X^{\eta, \epsilon} = \Psi^\epsilon(\sqrt{\epsilon}B + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle)$. Then

$$\begin{aligned}
X^{\eta, \epsilon}(x, t) = & x + \int_0^t b^\epsilon(X^{\eta, \epsilon}(x, s)) ds + \sqrt{\epsilon} \int_0^t \sigma^\epsilon(X^{\eta, \epsilon}(x, s)) dB_s \\
& + \int_0^t \sigma^\epsilon(X^{\eta, \epsilon}(x, s)) \eta_s d\langle B \rangle_s + \int_0^t h^\epsilon(X^{\eta, \epsilon}(x, s)) d\langle B \rangle_s.
\end{aligned} \tag{5.11}$$

and there exists a constant $M = M(\bar{\sigma})$ such that

$$\left| \int_0^t \sigma^\epsilon(X^{\eta, \epsilon}(x, s)) \eta_s d\langle B \rangle_s \right| \leq \left(\int_0^t |\sigma^\epsilon(X^{\eta, \epsilon}(x, s))|^2 ds \right)^{1/2} M r^{1/2}.$$

By the BDG inequality under G -expectation and Gronwall's equality, we can get that (cf. [14]) for $q \geq 2$, for any $m \geq 1$, there exists a constant $\beta = \beta(m, q, r, \bar{\sigma})$ such that

$$\sup_{\int_0^T |\eta_s|^2 ds \leq r} \sup_{\epsilon \in [0, 1]} \sup_{|x| \leq m} \mathbb{E}^G \left(\sup_{t \in [0, T]} |X^{\eta, \epsilon}(x, t)|^q \right) \leq \beta$$

and for any $x, y \in \mathbb{R}^q$, for any $s, t \in [0, T]$,

$$\sup_{\int_0^T |\eta_s|^2 ds \leq r} \sup_{\epsilon \in [0, 1]} \mathbb{E}^G (|X^{\eta, \epsilon}(x, t) - X^{\eta, \epsilon}(y, s)|^q) \leq \beta(|x - y|^q + |s - t|^{q/2}). \tag{5.12}$$

Set

$$\theta(\epsilon) = \sup_{x \in \mathbb{R}^p} \max \left\{ |b^\epsilon(x) - b(x)|, \|\sigma^\epsilon(x) - \sigma(x)\|_{HS}, \max_{1 \leq k \leq p} \|h^{\epsilon, k}(x) - h^k(x)\|_{HS} \right\}$$

and $Z^{\eta,\epsilon}(x, t) = X^{\eta,\epsilon}(x, t) - X^{\eta,0}(x, t)$. Then

$$\begin{aligned}
Z^{\eta,\epsilon}(x, t) = & \sqrt{\epsilon} \int_0^t \sigma^\epsilon(X^{\eta,\epsilon}(x, s)) dB_s + \int_0^t (b^\epsilon(X^{\eta,\epsilon}(x, s)) - b(X^{\eta,\epsilon}(x, s))) ds \\
& + \int_0^t (\sigma^\epsilon(X^{\eta,\epsilon}(x, s)) - \sigma(X^{\eta,\epsilon}(x, s))) \eta_s d\langle B \rangle_s \\
& + \int_0^t (h^\epsilon(X^{\eta,\epsilon}(x, s)) - h(X^{\eta,\epsilon}(x, s))) d\langle B \rangle_s \\
& + \int_0^t (b(X^{\eta,\epsilon}(x, s)) - b(X^{\eta,0}(x, s))) ds \\
& + \int_0^t (\sigma(X^{\eta,\epsilon}(x, s)) - \sigma(X^{\eta,0}(x, s))) \eta_s d\langle B \rangle_s \\
& + \int_0^t (h(X^{\eta,\epsilon}(x, s)) - h(X^{\eta,0}(x, s))) d\langle B \rangle_s,
\end{aligned}$$

and so for any $q \geq 2$, by the BDG inequality under G -expectation and Gronwall's equality, there exists a function $\gamma(\epsilon, \theta(\epsilon), q, r, \bar{\sigma})$ satisfying $\gamma(\epsilon, \theta(\epsilon), q, r, \bar{\sigma}) \rightarrow 0$ as $\epsilon \rightarrow 0$ such that (cf. [14])

$$\sup_{\int_0^T |\eta_s|^2 ds \leq r} \sup_{|x| \leq m} \bar{\mathbb{E}}^G \left(\sup_{t \in [0, T]} |Z^{\eta,\epsilon}(x, t)|^q \right) \leq \gamma(\epsilon, \theta(\epsilon), q, r, \bar{\sigma}),$$

which yields that

$$\lim_{\epsilon \rightarrow 0} \sup_{\int_0^T |\eta_s|^2 ds \leq r} \sup_{|x| \leq m} \bar{\mathbb{E}}^G \left(\sup_{t \in [0, T]} |Z^{\eta,\epsilon}(x, t)|^q \right) = 0. \quad (5.13)$$

Finally, by the below Lemma 5.4, (5.10) is a consequence of (5.12) and (5.13). \square

For given $N \geq 1$, for each $f \in \mathbb{H}$, $g \in \mathbb{A}$, let $\Psi^{(N)}(f, g) \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)$ be defined by

$$\begin{aligned}
& \Psi^{(N)}(f, g)(x, t) \\
= & x + \sum_{k=1}^N b \left(\Psi^{(N)}(f, g) \left(x, \frac{(k-1)T}{N} \right) \right) \left(\frac{kT}{N} \wedge t - \frac{(k-1)T}{N} \wedge t \right) \\
& + \sum_{k=1}^N \sigma \left(\Psi^{(N)}(f, g) \left(x, \frac{(k-1)T}{N} \right) \right) \left(f \left(\frac{kT}{N} \wedge t \right) - f \left(\frac{(k-1)T}{N} \wedge t \right) \right) \\
& + \sum_{k=1}^N h \left(\Psi^{(N)}(f, g) \left(x, \frac{(k-1)T}{N} \right) \right) \left(g \left(\frac{kT}{N} \wedge t \right) - g \left(\frac{(k-1)T}{N} \wedge t \right) \right).
\end{aligned}$$

Then it is obvious that for any $N \geq 1$, $(\mathbb{H} \times \mathbb{A}, \rho_{HG}) \ni (f, g) \rightarrow \Psi^{(N)}(g)$ is continuous and

$$\begin{aligned} \Psi^{(N)}(f, g)(x, t) = & x + \int_0^t b(\Psi^{(N)}(f, g)(x, \pi_N(s))) ds \\ & + \int_0^t \sigma(\Psi^{(N)}(f, g)(x, \pi_N(s))) df(s) \\ & + \int_0^t h(\Psi^{(N)}(f, g)(x, \pi_N(s))) dg(s) \end{aligned}$$

where $\pi_N(s) = \frac{(k-1)T}{N}$, for $s \in [(k-1)T/N, kT/N)$, $k = 1, \dots, N$.

Lemma 5.3. *Assume that (H1) and (H2) hold. Then for any $l \in (0, \infty)$,*

$$\lim_{N \rightarrow \infty} \sup_{\|f\|_H \leq l, g \in \mathbb{A}} \rho(\Psi(f, g), \Psi^{(N)}(f, g)) = 0.$$

Proof. Firstly, by the Lipschitz condition, there exists a constant $L_1 \in (0, \infty)$ such that for any $x \in \mathbb{R}^p$, $t \in [0, T]$, $f \in \mathbb{H}$, $g \in \mathbb{A}$,

$$|\Psi(f, g)(x, t)| \leq |x| + L_1 \int_0^t (1 + |\Psi(f, g)(x, s)|) (1 + |f'(s)| + \|g'(s)\|_{HS}) ds.$$

Therefore, by Gronwall's inequality, for any $m \geq 1$,

$$\bar{M}_m := \sup_{\|f\|_H \leq l, g \in \mathbb{A}} \sup_{|x| \leq m, t \in [0, T]} |\Psi(f, g)(x, t)| < \infty.$$

Furthermore, there exist positive constants L_2, L_3 such that for any $|x| \leq m$, $t \in [0, T]$, $\|f\|_H \leq l$, $g \in \mathbb{A}$,

$$\begin{aligned} & |\Psi(f, g)(x, \pi_N(t)) - \Psi(f, g)(x, t)| \\ & \leq L_2 \max_{1 \leq k \leq N} \max_{t \in [(k-1)T/N, kT/N]} \left(\int_{(k-1)T/N}^t |f'(s)| + \|g'(s)\|_{HS} ds \right) \leq \frac{L_3}{\sqrt{N}}. \end{aligned}$$

Therefore, by

$$\begin{aligned}
& \left| \Psi^{(N)}(f, g)(x, t) - \Psi(f, g)(x, t) \right| \\
& \leq \int_0^t \left| b(\Psi^{(N)}(f, g)(x, \pi_N(s))) - b(\Psi(f, g)(x, \pi_N(s))) \right| ds \\
& \quad + \int_0^t \left| b(\Psi(f, g)(x, s)) - b(\Psi(f, g)(x, \pi_N(s))) \right| ds \\
& \quad + \int_0^t \left| (\sigma(\Psi^{(N)}(f, g)(x, \pi_N(s))) - \sigma(\Psi(f, g)(x, \pi_N(s)))) f'(s) \right| ds \\
& \quad + \int_0^t \left| (\sigma(\Psi(f, g)(x, s)) - \sigma(\Psi(f, g)(x, \pi_N(s)))) f'(s) \right| ds \\
& \quad + \int_0^t \left| (h(\Psi^{(N)}(f, g)(x, \pi_N(s))) - h(\Psi(f, g)(x, \pi_N(s)))) g'(s) \right| ds \\
& \quad + \int_0^t \left| (h(\Psi(f, g)(x, s)) - h(\Psi(f, g)(x, \pi_N(s)))) g'(s) \right| ds,
\end{aligned}$$

there exist positive constants L_4, L_5 such that for any $|x| \leq m, t \in [0, T], \|f\|_H \leq l, g \in \mathbb{A}$,

$$\begin{aligned}
& \max_{s \in [0, t]} \left| \Psi^{(N)}(f, g)(x, s) - \Psi(f, g)(x, s) \right| \\
& \leq \frac{L_4}{\sqrt{N}} + L_5 \int_0^t \max_{u \in [0, s]} \left| \Psi^{(N)}(f, g)(x, u) - \Psi(f, g)(x, u) \right| (1 + |f'(s)| + \|g'(s)\|_{HS}) ds.
\end{aligned}$$

Therefore, by Gronwall lemma, we obtain that for any $m \geq 1$,

$$\lim_{N \rightarrow \infty} \sup_{|x| \leq m, t \in [0, T], \|f\|_H \leq l, g \in \mathbb{A}} \left| \Psi^{(N)}(f, g)(x, t) - \Psi(f, g)(x, t) \right| = 0.$$

□

Lemma 5.4. *Let $T > 0$ and let $\{Y_{\lambda, \epsilon} = \{Y_{\lambda, \epsilon}(t), t \in [0, T]^m\}; \epsilon \in [0, 1], \lambda \in \Lambda\}$ be a family of \mathbb{R}^p -valued continuous processes such that $Y_{\lambda, \epsilon}(t)$ is quasi-continuous for all λ, ϵ and t . Assume that there exists constants $L \in (0, +\infty), q > 0$ and $\kappa > 0$ such that*

$$\sup_{\lambda \in \Lambda, \epsilon \in [0, 1]} \mathbb{E}^G(|Y_{\lambda, \epsilon}(t) - Y_{\lambda, \epsilon}(s)|^q) \leq C|t - s|^{m+\kappa}, \quad s, t \in [0, T]^m. \quad (5.14)$$

Then

$$\sup_{\lambda \in \Lambda, \epsilon \in [0, 1]} \mathbb{E}^G \left(\left(\sup_{s \neq t} \frac{|Y_{\lambda, \epsilon}(t) - Y_{\lambda, \epsilon}(s)|}{|t - s|^\alpha} \right)^q \right) < \infty, \quad (5.15)$$

for every $\alpha \in [0, \kappa/q)$. As a consequence, $\{\{Y_{\lambda, \epsilon}(t), t \in [0, T]^m\}; \epsilon \in [0, 1], \lambda \in \Lambda\}$ is tight under \mathbb{E}^G , i.e., for any $\delta > 0$, there exists a compact $K_\delta \subset C([0, T]^m, \mathbb{R}^p)$ such that

$$\sup_{\lambda \in \Lambda, \epsilon \in [0, 1]} c^G(Y_{\lambda, \epsilon} \in K_\delta^c) < \delta. \quad (5.16)$$

Furthermore, if for $t \in [0, T]^m$ and any $\delta > 0$,

$$\limsup_{\epsilon \rightarrow 0} c^G (|Y_{\lambda, \epsilon}(t) - Y_\lambda(t)| \geq \delta) = 0, \quad (5.17)$$

where $Y_\lambda(t) := Y_{\lambda, 0}(t)$, then $Y_{\lambda, \epsilon}$ converges uniformly to Y_λ in distribution under \mathbb{E}^G , i.e., for any $\Phi \in C_b(C([0, T]^m, \mathbb{R}^p))$,

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E}^G (|\Phi(Y_{\lambda, \epsilon}) - \Phi(Y_\lambda)|) = 0. \quad (5.18)$$

Proof. First, from the proof of the Kolmogorov criterion under G -expectation (cf. Theorem 1.36, Chapter VI in [22]), we can obtain (5.15). Since for each $\alpha \in (0, \kappa/q)$,

$$\left\{ y \in C([0, T]^m, \mathbb{R}^p); \sup_{s \neq t} \frac{|y(t) - y(s)|}{|t - s|^\alpha} \leq r \right\}$$

is compact subset for any $r \in (0, \infty)$, by Chebyshev's inequality and (5.15), for any $\delta > 0$, there exists a compact $K_\delta \subset C([0, T]^m, \mathbb{R}^p)$ such that (5.16) holds.

If for each $t \in [0, T]^m$ and $\delta > 0$, (5.17) holds. Take $\alpha \in (0, \kappa/q)$. For any $\delta > 0$, choose $r = r(\delta) \in (0, \infty)$ such that

$$\sup_{\lambda \in \Lambda, \epsilon \in [0, 1]} c^G (Y_{\lambda, \epsilon}(t) \in K_r^c) < \delta,$$

$$\text{where } K_r = \left\{ y \in C([0, T]^m, \mathbb{R}^p); \sup_{s \neq t} \frac{|y(t) - y(s)|}{|t - s|^\alpha} \leq r \right\}.$$

By continuity of Φ and compactness of K_r , there exists $\zeta > 0$ such that for any $\psi, \varphi \in C([0, T]^m, \mathbb{R}^p) \cap K_r$ with $\|\psi - \varphi\| \leq \zeta$, $|\Phi(\psi) - \Phi(\varphi)| < \delta$.

By the definition of K_r , there exist $l \geq 1$, $\tau \in (0, (\zeta/3)^{1/\alpha}/r)$ and t_1, \dots, t_l such that $[0, T]^m = \cup_{i=1}^l U(t_i, \tau)$, and

$$\sup_{\lambda \in \Lambda} c^G \left(\max_{1 \leq i \leq l} \sup_{t \in U(t_i, \tau)} |Y_\lambda(t) - Y_\lambda(t_i)| \geq \zeta/3, Y_\lambda \in K_r \right) = 0,$$

where $U(t_i, \tau) = \{t \in [0, T]^m; |t - t_i| < \tau\}$.

By (5.17), there exists ϵ_0 such that for all $\epsilon \in (0, \epsilon_0)$,

$$\max_{1 \leq i \leq l} \sup_{\lambda \in \Lambda} c^G (|Y_{\lambda, \epsilon}(t_i) - Y_\lambda(t_i)| \geq \zeta/3) < \delta/3,$$

By triangle inequality

$$|Y_{\lambda, \epsilon}(t) - Y_\lambda(t)| \leq |Y_{\lambda, \epsilon}(t) - Y_{\lambda, \epsilon}(t_i)| + |Y_{\lambda, \epsilon}(t_i) - Y_\lambda(t_i)| + |Y_\lambda(t) - Y_\lambda(t_i)|,$$

we have that for all $\epsilon \in (0, \epsilon_0)$,

$$\sup_{\lambda \in \Lambda} c^G \left(Y_{\lambda, \epsilon} \in K_r, Y_\lambda \in K_r, \sup_{t \in [0, T]^m} |Y_{\lambda, \epsilon}(t) - Y_\lambda(t)| \geq \zeta \right) < \delta.$$

Therefore, for all $\epsilon \in (0, \epsilon_0)$, $\sup_{\lambda \in \Lambda} \mathbb{E}^G (|\Phi(Y_{\lambda, \epsilon}) - \Phi(Y_\lambda)|) \leq (1 + 3M)\delta$, where $M := \sup_{y \in C([0, T]^m, \mathbb{R}^p)} |\Phi(y)|$. This yields (5.18). \square

REFERENCES

- [1] Baldi, P. and Sanz-solé, M.: Modulus of Continuity for Stochastic Flows, *Barcelona Seminar on Stochastic Analysis*, (Nualart, D. and San-solé, M. ed.) Birkhäuser, Prog. Probab. 32(1993), 1-20.
- [2] Ben Arous, G. and Castell, F. Flow decomposition and large deviations. *J. Funct. Anal.* 140 (1995), 23–67.
- [3] Borell, C., Diffusion equations and geometric inequalities, *Potential Anal.*, 12 (2000) 49–71.
- [4] Boué, M. and Dupuis, P., A variational representation for certain functionals of Brownian motion. *Ann. Probab.*, 26(1998), 1641-1659.
- [5] Budhiraja, A., Dupuis, P. and Maroulas, V., Large deviations for infinite dimensional stochastic dynamical systems. *Ann. Probab.*, 36(2008), 1390–1420.
- [6] Budhiraja, A., Dupuis, P. and Maroulas, V., Large deviations for stochastic flows of diffeomorphisms. *Bernoulli*, 16(2010), 234–257.
- [7] Chen, Z. J. and Xiong, J., Large deviation principle for diffusion processes under a sublinear expectation. Preprint 2010.
- [8] Dembo, A., Zeitouni, D., *Large Deviations Techniques and Applications*, Springer-Verlag, 1998.
- [9] Denis, L., Martini, C., A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *Ann. Appl. Probab.* 16(2)(2006) 827-852.
- [10] Denis, L., Hu, M. S. and Peng, S., Function spaces and capacity related to a sublinear expectation: application to G-Brownian Motion Paths. *Potential Analysis*, 34 (2011), 139–161.
- [11] Dupuis, P. and Ellis, R. S., *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley, New-York, 1997.
- [12] Freidlin, M. I. and Wentzell, A. D., *Random Perturbations of Dynamical Systems*. Springer-Verlag, New York, 1984.
- [13] Gao, F. Q. and Ren, J. G.: Large deviations for stochastic flows and their applications. *Science in China*, 44(2001), 1016-1033.
- [14] Gao, F. Q., Pathwise properties and homeomorphic flows for stochastic differential equations driven by G -Brownian motion. *Stoch. Proc. Appl.*, 119(2009), 3356-3382.
- [15] Gao, F. Q. and Jiang, H., Large Deviations for Stochastic Differential Equations Driven by G -Brownian Motion. *Stoch. Proc. Appl.*, 120 (2010), 2212–2240.
- [16] Kowalski, E., Bernstein Polynomials and Brownian Motion. *The American Mathematical Monthly*, 113 (2006), 865–886.
- [17] Krylov N. K., *Nonlinear elliptic and Parabolic Equations of the Second Order*, Kluwer, 1987.
- [18] Millet, A., Nualart, D. and Sanz-Solé, M., Large deviations for a class of anticipating stochastic differential equations. *Ann. Probab.* 20(1992), 1902–1931.
- [19] Osuka, E. Girsanov’s formula for G -Brownian motion, arXiv:1106.2387, 2011.
- [20] Peng, S. G.: G -Expectation, G -Brownian motion and related stochastic calculus of Itô’s type, in: *Proceedings of the 2005 Abel Symposium 2*, Edit. Benth et. al. 541–567, Springer-Verlag, 2006.
- [21] Peng, S. G., Multi-dimensional G -Brownian Motion and related stochastic calculus under G -expectation. *Stoch. Proc. Appl.*, 118(2008), 2223–2253.
- [22] Peng, S. G, Nonlinear Expectations and Stochastic Calculus under Uncertainty. arXiv:math.PR/1002.4546, 2010.
- [23] Ren, J. and Zhang, X., Freidlin-Wentzell’s large deviations for homeomorphism flows of non-Lipschitz SDEs. *Bull. Sci. Math.*, 129(2005), 643–655.
- [24] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*. Grund. Math. Wiss. 293, Springer-Verlag, 1998.

- [25] Soner, H. M., Touzi, N. and Zhang J. F., Martingale representation theorem for the G -expectation. *Stoch. Proc. Appl.*, 121(2011), 265–287.
- [26] Soner, H. M., Touzi, N. and Zhang J. F., Wellposedness of second order backward SDEs, arXiv:1003.6053, (2010). *Probability Theory and Related Fields*, forthcoming. arXiv:1003.6053, (2010).
- [27] Varadhan, S. R. S. *Large Deviations and Applications*, SIAM, Philadelphia, 1984.
- [28] Xu, J. and Zhang, B. Martingale characterization of G -Brownian motion. *Stoch. Proc. Appl.*, 119 (2009), 232–248.
- [29] Xu, J., Shang, H., Zhang, B., A Girsanov type theorem under G -framework, *Stoch. Anal. Appl.* 29 (2011) 386–406.

FUQING GAO, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, 430072
WUHAN, CHINA

E-mail address: fqgao@whu.edu.cn